Digital Communications
Random Processes and Spectral Analysis

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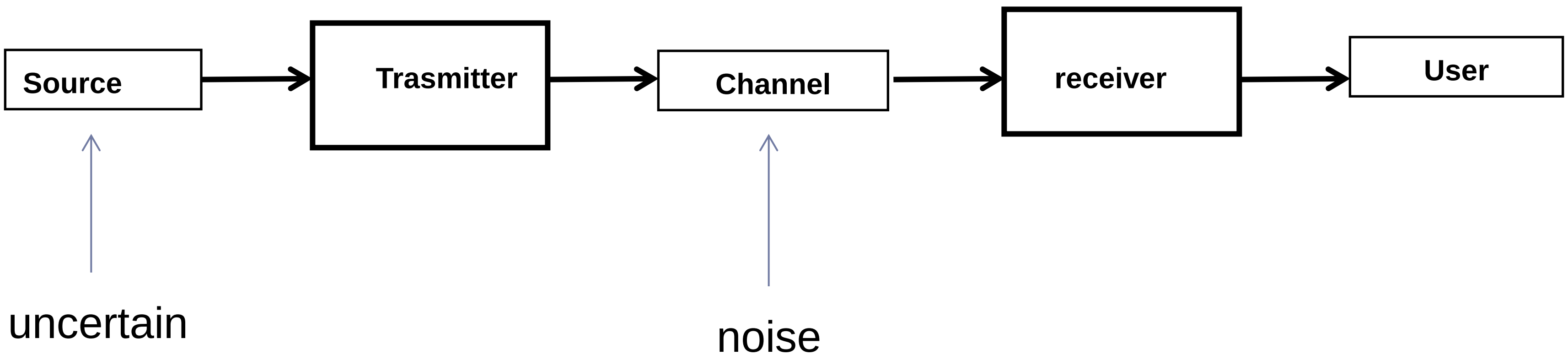
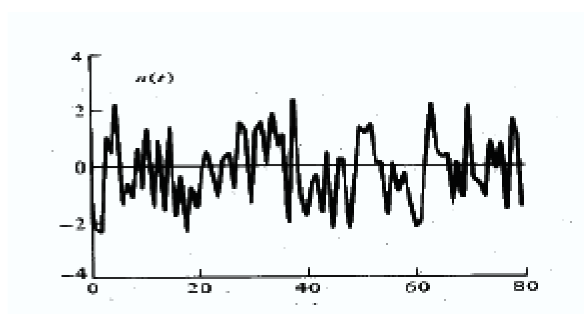
DIGITAL COMUNICATION SYSTEM
Random Process





DIGITAL COMUNICATION SYSTEM
Deterministic and Random Signals

A signal $x(t)$ is DETERMINISTIC if it is a known function of t,
that is, given t, the value of $x(t)$ is exactly known a priori.
For instance, the values of $x(t)=\cos(2\ t)$ are known for each t.

A signal $x(t)$ is a RANDOM PROCESS if, given a time t,
the value of $x(t)$ can be only characterized statistically,
i.e. it is a random variable which is characterized by some probability density function.

An example of a random process is the thermal noise in electronic devices.

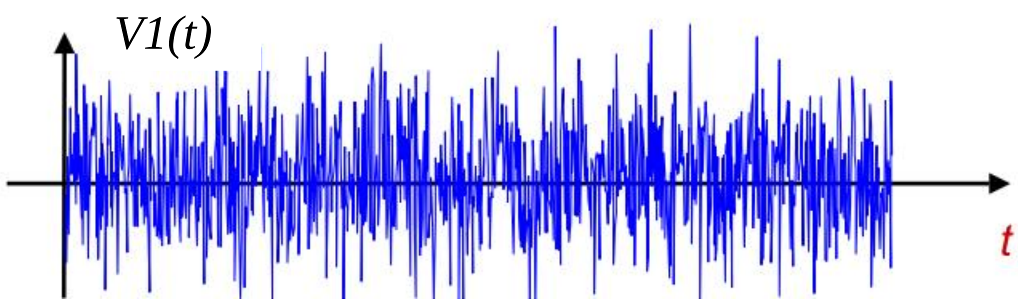
The value of this signal is **not known «a priori»** but it can be known only when it has been **measured**.

Example of Random Process: Thermal Noise

Let us consider the weak electric voltage at both ends of a resistor.

It is a function of time and it is caused by the chaotic movements of electrons due to the temperature that is not the absolute zero.

Once I have measured the voltage we get the following signal $V1(t)$

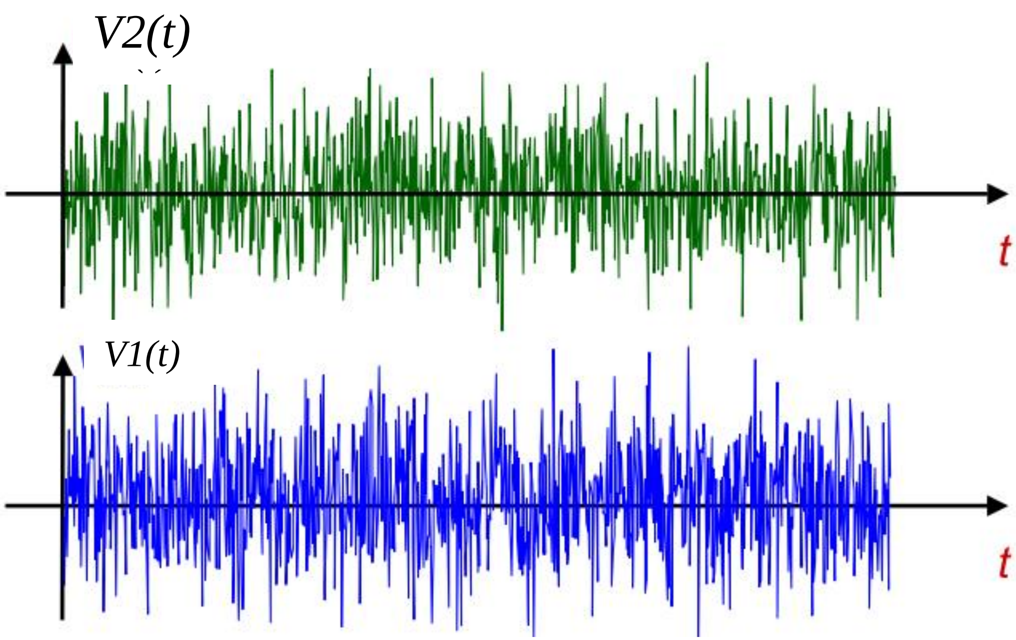


which can be considered deterministic (after it has been measured).

DIGITAL COMUNICATION SYSTEM
Deterministic and Random Signals

Example of Random Process: Thermal Noise

Let us take another resistor that is identical to the first one, same temperature, we make another measurements and we get another evolution over time of the voltage,let us call it *V2(t)*. I get another time evolution even if I use the same resistor but 10 minutes later!!!!



V2(t) has similar characteristics but is different from *V1(t)* as electrons move randomly, and an independent way, in the two resistors or in the same resistors but later in time!

Example of Random Process: Thermal Noise

To design an electronic device taking into account this noise, does it help to know the time evolution of $V1(t)$ or $V2(t)$ knowing that if I use another resistor (even if identical) the time evolution is different? NO!

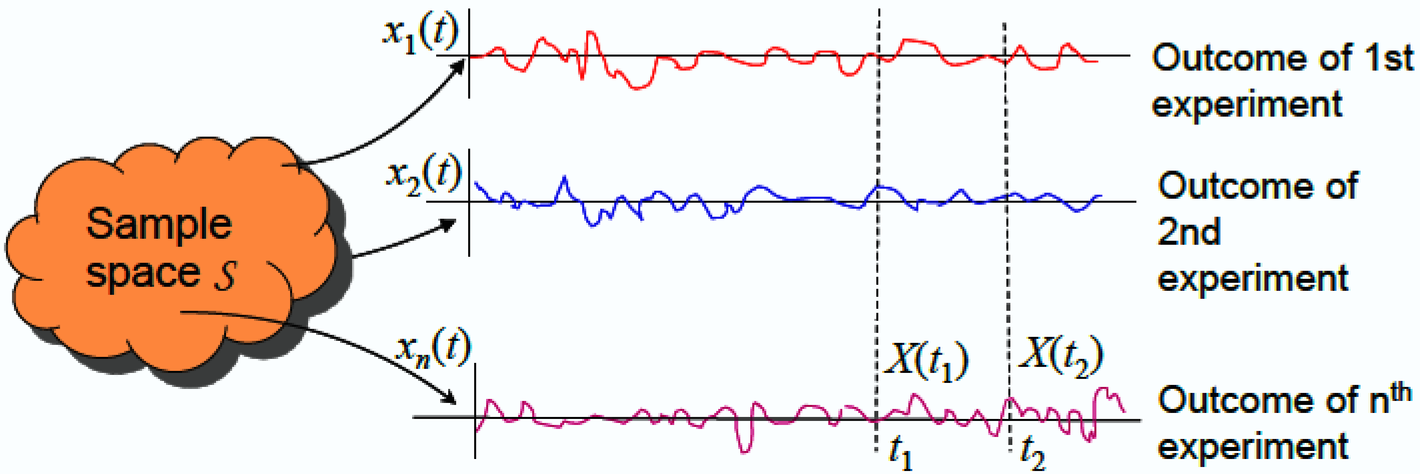
What is is useful is to describe the features of the noise voltage that are *COMMON* to all resistors that are identical and that are at the same temperature.

In this way, if I use this type of resistor (and at a given temperature) inside a device, I can say, for instance, what is the probability to get a given voltage within a range of values, or what is the noise power.

Therefore, with random process we must use the «tools» provided by the probability theory.

DIGITAL COMUNICATION SYSTEM
Random Processes

- $X(t)$: random process
- $x_n(t)$: sample function of the random process
- $X(t_1), X(t_2), \ldots$: values of the random process at $t_1, t_2, \ldots,$



Random Processes

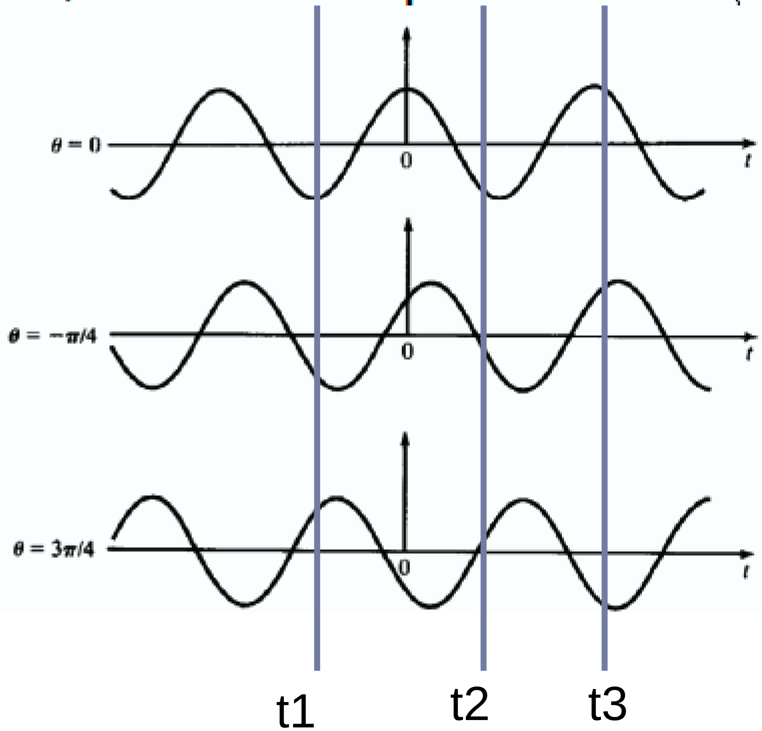
About a random process, it is useful to know the <u>characteristics that are common to all sample functions.</u>

A random process is COMPLETELY described by its <u>joint probability density functions of all orders and for any time instants</u>.

Often, this information is not available and from some type of random process it is enough to know only low order probability density functions.

More often, for describing a random process it is used the probability density function (pdf) of its amplitudes and the autocorrelation (and in some specific random process, these two information are enough!).

- Uniformly choose a phase Θ between $(0, 2\pi)$ and generate a sinusoid with a fixed amplitude and frequency but with a random phase Θ.
- In this case, the random process is $X(t) = A\cos(2\pi f_0 t + \Theta)$



- An infinite collection of random variables specified at time *t*

$$\{X(t_1), X(t_2), ..., X(t_n)\}$$

- Joint pdf

$$f\left(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n\right) \quad \forall n$$

- $f(x;t)$ = first order density of $X(t)$

- Mean

$$E[X(t_0)] = E[X(t = t_0)] = \int_{-\infty}^{\infty} x f_X(x; t_0) = \overline{X}(t_0)$$

- Variance

$$E[|X(t_0) - \overline{X}(t_0)|^2] = \sigma_X^2(t_0)$$

- $f(x_1, x_2; t_1, t_2)$ = second-order density of $X(t)$

- Auto-correlation function (correlation within a process):

$$R_X(t_1, t_2) = E\big[X(t_1)X(t_2)\big]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2; t_1, t_2)\,dx_1 dx_2$$

- Consider $X(t) = A\cos(2\pi f t + \theta)$, where θ is uniform in $(-\pi, \pi)$
- Mean: $E[X(t)] = \int_{-\pi}^{\pi} A\cos(2\pi f t + \theta)\frac{1}{2\pi}d\theta = 0$

of of the uniform distribution between (-π,π)

- Auto-correlation: Let $t_1 = t$ and $t_2 = t + \tau$

$$E[X(t_1)X(t_2)] = E[A\cos(2\pi f t + \theta)A\cos(2\pi f(t+\tau) + \theta)]$$

$$= \tfrac{A^2}{2}E[\cos(4\pi f t + 2\pi f t + 2\theta) + \cos(2\pi f \tau)]$$

$$= \tfrac{A^2}{2}\int_{-\pi}^{\pi}\tfrac{1}{2\pi}\cos(4\pi f t + 2\pi f \tau + 2\theta)d\theta + \tfrac{A^2}{2}\int_{-\pi}^{\pi}\tfrac{1}{2\pi}\cos(2\pi f \tau)d\theta$$

$$= 0 + \tfrac{A^2}{2}\cos(2\pi f \tau)$$

$$\Rightarrow R_X(t;\tau) = \frac{A^2}{2}\cos(2\pi f \tau)$$

- Consider $Y(t) = B \cos w_c t$, where $B \sim \mathcal{N}(0, b^2)$
- Find its mean and auto-correlation function

- A stochastic process is said to be stationary if for any n and τ:

$$f(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = f(x_1, x_2, \ldots, x_n; t_1+\tau, t_2+\tau, \ldots, t_n+\tau) \qquad (1)$$

- First-order statistics is independent of t

$$\Rightarrow \quad E\{X(t)\} = \int_{-\infty}^{\infty} x f_X(x)dx = m_X \qquad (2)$$

- Second-order statistics only depends on the gap $t_2 - t_1$

$$\Rightarrow \quad R_X(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2, t_2 - t_1) dx_1 dx_2$$

$$= R_X(t_2 - t_1) = R_X(\tau), \quad \text{where } \tau = t_2 - t_1 \qquad (3)$$

- A random process is said to be **WSS** when:

$$E\{X(t)\} = \int_{-\infty}^{\infty} x f_X(x) dx = m_X \qquad (2)$$

$$R_X(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1,x_2,t_2-t_1) dx_1 dx_2$$
$$= R_X(t_2-t_1) = R_X(\tau), \quad \text{where } \tau = t_2 - t_1 \qquad (3)$$

- A random process is **strictly stationary** when:

$$f(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n)$$
$$= f(x_1, x_2, \ldots, x_n; t_1+\tau, t_2+\tau, \ldots, t_n+\tau) \quad \forall n, \tau \qquad (1)$$

- Example 1: $X(t) = A\cos(2\pi f t + \theta)$,where $\theta \sim U(-\pi, \pi)$

From the previous example:

$$E\left[X(t)\right] = 0$$

$$R_X(t_1, t_2) = \frac{A^2}{2}\cos\left(2\pi f(t_2 - t_1)\right)$$

$\Longrightarrow$ X(t) is WSS

- Example 2: $Y(t) = B\cos w_c t$, where $B \sim \mathcal{N}(0, b^2)$

Is Y(t) WSS?

PSD of WSS Process

- **Wiener-Khinchin theorem**

$$S_X(f) \leftrightarrow R_X(\tau) \quad \begin{cases} R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)\exp(j2\pi f\tau)df \\ S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\exp(-j2\pi f\tau)d\tau \end{cases}$$

- Property:

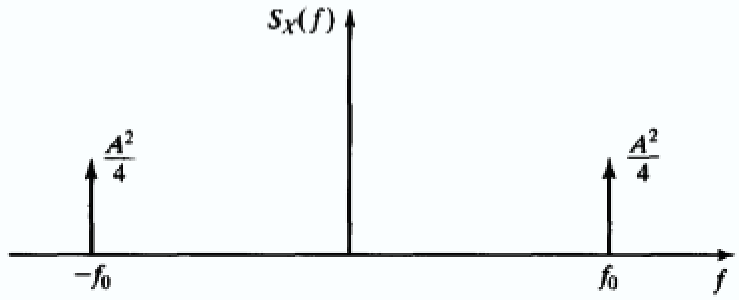
$$R_X(0) = \int_{-\infty}^{\infty} S_X(f)df = \text{total power}$$

- For the random process $X(t) = A\cos(2\pi f t + \theta)$
- We had

$$\Rightarrow R_X(t;\tau) = \frac{A^2}{2}\cos(2\pi f \tau)$$

- Hence

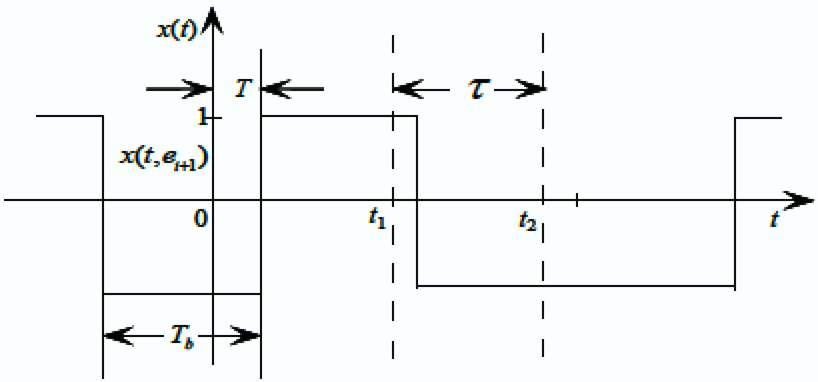
$$S_X(f) = \frac{A^2}{4}\left[\delta(f - f_0) + \delta(f + f_0)\right]$$



- Given a binary random signal

$$X(t) = \sum_n a_n p(t - nT_b - T)$$

  - $p(t)$ is a rectangular pulse shaping function with width T_b
  - a_n is a random variable that takes +1 or -1 with equal probability, and it is independent for different n
  - T is a random time delay uniformly distributed within $[0, T_b]$
  - A typical sample function of X(t) is



DIGITAL COMUNICATION SYSTEM
Random Processes

$$R(t_i, t_i + \tau) = E[X(t_i)X(t_i + \tau)]$$

First case $|\tau| > T_b$

$$R(t_i, t_i + \tau) = E[X(t_i)X(t_i + \tau)] = E[X(t_i)]E[X(t_i + \tau)] = 0$$

Second case $|\tau| < T_b$

The random variable $X(t_i), X(t_i + \tau)$ occur in the same pulse interval if and only if the delay satisfies the condition $T + |\tau| < T_b \Rightarrow T < T_b - |\tau|$

$$E[X(t_i)X(t_i + \tau)/T] = \begin{cases} A^2, & \mathrm{T} < \mathrm{T_b} - |\tau| \\ 0, & \text{elsewhere} \end{cases}$$

DIGITAL COMUNICATION SYSTEM
Random Processes

Averaging the resul over all positive values of T, we get

$$E[X(t_i)X(t_i+\tau)] = \int_0^{T_b-|\tau|} A^2 f_T(T)\,dT = \int_0^{T_b-|\tau|} A^2 \frac{1}{T_b}\,dT = A^2\left(1-\frac{|\tau|}{T_b}\right)$$

It is a WSS random process

- Find the autocorrelation function

$$R_X(\tau) = 1 - \frac{|\tau|}{T_b}, \quad -T_b < \tau < T_b$$
$$= 0, \quad otherwise$$

- Find its power spectral density

$$S_X(f) = T_b \left(\frac{\sin \pi f T_b}{\pi f T_b} \right)^2$$

DIGITAL COMUNICATION SYSTEM
Random Processes through n LTI system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Stationary random process

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Expected value $\mu_y(t) = E[y(t)] = \int_{-\infty}^{\infty} h(\tau)\underbrace{E[x(t-\tau)]}_{\mu_x}d\tau = \mu_x H(0)$

$$R_y(t,\tau) = E\left[\underbrace{\int h(\tau_1)x(t-\tau_1)\,d\tau_1}_{y(t)} \underbrace{\int h(\tau_2)x(t-\tau_2)\,d\tau_2}_{y(v)}\right] =$$

$\tau = t - v$

$$= \int h(\tau_1) \int h(\tau_2)\underbrace{E[x(t-\tau_1)x(v-\tau_2)]}_{R_x(t-\tau_1-v+\tau_2) = R_x(\tau-\tau_1+\tau_2)}\,d\tau_1 d\tau_2$$ It does not depend on t

The output is a STATIONARY random process

DIGITAL COMUNICATION SYSTEM
Random Processes through n LTI system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Stationary random process

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$S_y(f) = \int df \left[\int h(\tau_1) \int h(\tau_2)\, R_x(\tau - \tau_1 + \tau_2) \right] e^{-i2\pi f\tau} d\tau$$

$$\tau_0 = \tau - \tau_1 - \tau_2$$

$$\int df \left[\int h(\tau_1) \int h(\tau_2)\, R_x(\tau_0) \right] e^{-i2\pi f(\tau_0 + \tau_1 - \tau_2)} d\tau_0 =$$

$$\int h(\tau_1) e^{-i2\pi f\tau_1} d\tau_1 \int h(\tau_2) e^{+i2\pi f\tau_2} d\tau_2 \int R_x(\tau_0) e^{-i2\pi f\tau_0} d\tau_0 =$$

$$\boxed{H(f)H^*(f)S_x(f) = |H(f)|^2 S_x(f)}$$

DIGITAL COMUNICATION SYSTEM
Ergodicity

Time average over an observation interval T

$$\mu_x(T) = \frac{1}{2T}\int_{-T}^{T} x(t)\, dt$$

It is a random varialbe

Being the random process stationary

$$E[\mu_x(T)] = \frac{1}{2T}\int_{-T}^{T} E[x(t)]\, dt = \frac{1}{2T}\int_{-T}^{T} \mu_X\, dt = \mu_X$$

The stationary random process is ERGODIC if it verifies the conditions:

$$\lim_{T\to\infty} \mu_x(T) = \mu_X$$

$$\lim_{T\to\infty} \operatorname{var}[\mu_x(T)] = 0$$

$$\lim_{T\to\infty} R_x(\tau, T) = R_X(\tau)$$

$$\lim_{T\to\infty} \operatorname{var}[R_x(\tau, T)] = 0$$