Digital Communications

Signals, Random Processes and Spectral Analisys

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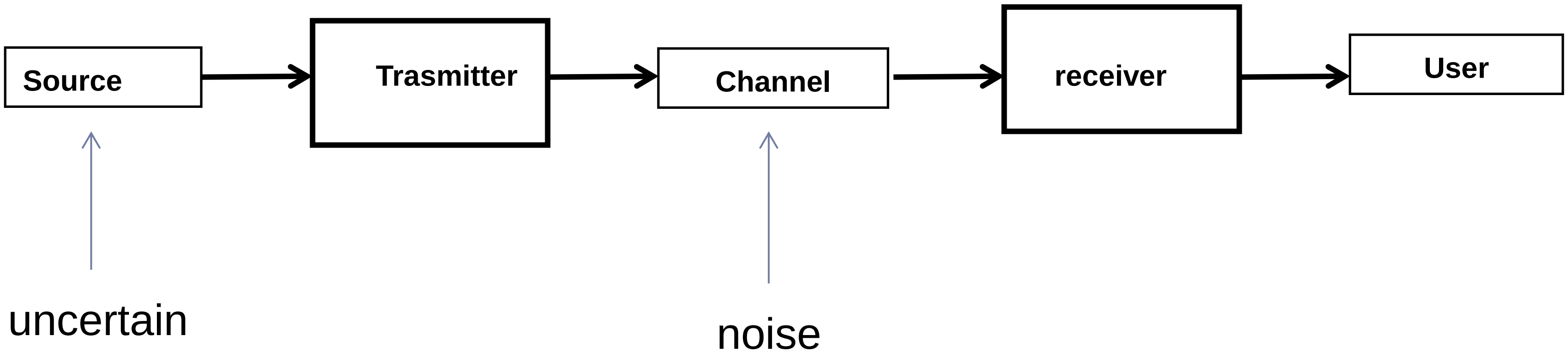
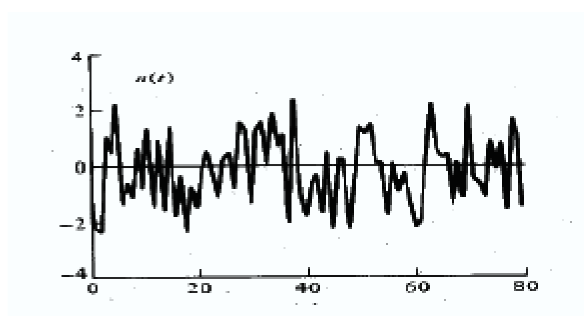
a.a. 2017-2018

DIGITAL COMUNICATION SYSTEM
Signals





DIGITAL COMUNICATION SYSTEM
Elements of Statistics

- Consider two events A and B
- Conditional probability P(A|B)
- Joint probability $P(AB) = P(A \cap B)$

$$P(AB) = P(B)P(A \mid B) = P(A)P(B \mid A)$$

- A and B are said statistically independent iff

$$P(AB) = P(A)P(B) \Rightarrow \begin{cases} P(A|B) = P(A) \\ P(B|A) = P(B) \end{cases}$$

- Let $A_j, \quad j = 1, 2, \ldots, n$ be mutually exclusive events with $A_i \cap A_j = \emptyset, \ \forall i \neq j$

- For any event B, we have

$$\begin{aligned} P(B) &= \sum_{j=1}^{n} P(B \cap A_j) \\ &= \sum_{j=1}^{n} P(B|A_j)P(A_j) \end{aligned}$$

Bayes's Theorem

- Let $A_i, \quad i = 1, 2, \ldots, n$ be mutually exclusive events such that $\cup_{i=1}^{n} A_i = S$ and B is an arbitrary event with nonzero probability. Then

$$\boxed{\begin{aligned} P(A_i|B) &= \frac{P(A_i, B)}{P(B)} \\ &= \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{n} P(B|A_j)P(A_j)} \end{aligned}}$$

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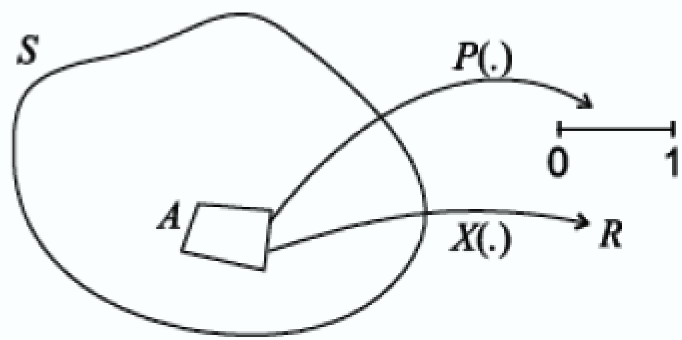
Elements of Statistics
Random Variables

- A r.v. is a mapping from the sample space S to the set of real numbers.

$$X(.): A \subset S \rightarrow x \in R \quad X(A) = x$$



- A r.v. may be
  - Discrete-valued: range is finite (e.g. {0,1}), or countable infinite (e.g. {1,2,3 …})
  - Continuous-valued: range is uncountable infinite (e.g. $\mathcal{R}$)

DIGITAL COMUNICATION SYSTEM
Elements of Statistics
Random Variables

- The Cumulative distribution function (CDF of a r.v. X, is

$$F_X(x) \overset{\Delta}{=} P(X \le x)$$

- Key properties of CDF
  1. $0 \le F_X(x) \le 1$ with $F_X(-\infty) = 0$ and $F_X(\infty) = 1$
  2. $F_X(x)$ is a non-decreasing function of x
  3. F $(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1)$

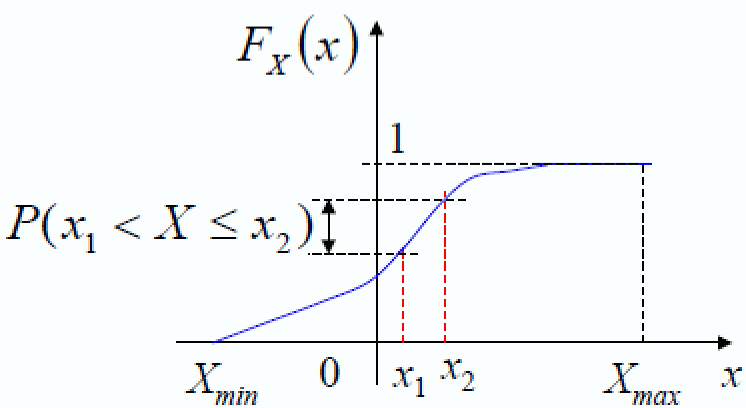
Random Variables

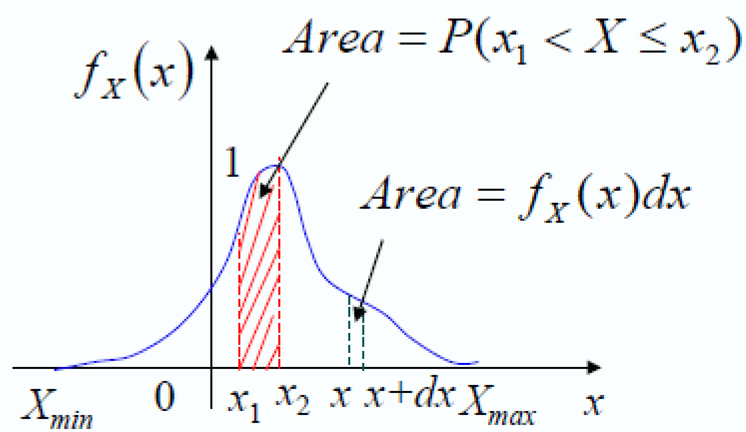
- The PDF, of a r.v. X, is defined as

$$f_X(x) \stackrel{\Delta}{=} \frac{d}{dx} F_X(x) \qquad \text{or} \qquad F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy$$

- Key properties of PDF

1. $p_X(x) \geq 0$
2. $\int_{-\infty}^{\infty} p_X(x)\,dx = 1$
3. $P(x_1 < X \leq x_2) = P_X(x_2) - P_X(x_1) = \int_{x_1}^{x_2} p_X(x)\,dx$





Bernoulli Distribution

- A discrete r.v taking two possible values, X = 1 or X = 0. with probability mass function (pmf)

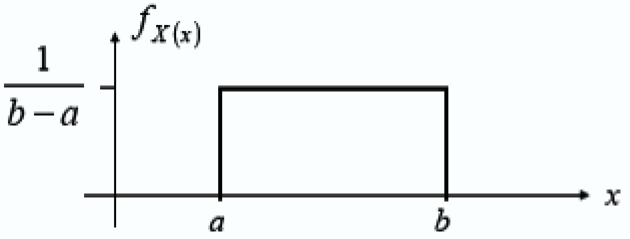
$$\begin{aligned} p(x) &= P(X = x) \\ &= \begin{cases} 1 - p & x = 0 \\ p & x = 1 \end{cases} \end{aligned}$$

- Often used to model binary data

Uniform Distribution

- A continuous r.v. taking values between *a* and *b* with equal probabilities
- The probability density function (pdf) is given by

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$



- The random phase of a sinusoid is often modeled as a uniform r.v. between 0 and 2π

Statistical Averages

- Consider a discrete r.v. which takes on the possible values $x_1, x_2, \ldots, x_M$ with respective probabilities $P_1, P_2, \ldots, P_M$.
- The mean or expected value of X is

$$m_X = E[X] = \sum_{i=1}^{M} x_i P_i$$

- If X is continuous, then

$$m_X = E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

- This is the first moment of X.

Statistical Averages

- The *n^{th} moment* of X

$$E\left[X^n\right] = \int_{-\infty}^{\infty} x^n p_X(x)dx$$

- Let n = 2, we have the mean-square value of X

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 p_X(x)dx$$

Statistical Averages

- *n*-th Central moment is

$$E\left[(X-m_X)^n\right]=\int_{-\infty}^{\infty}(x-m_x)^n f_X(x)dx$$

- At n=2, we have the variance

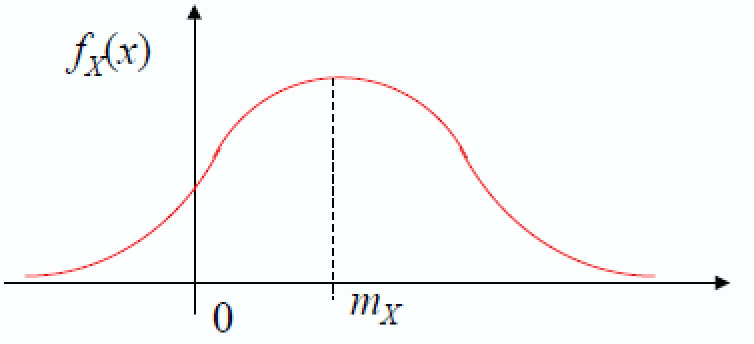
$$\begin{aligned}\sigma_X^2 &= E\left[(X-m_X)^2\right] \\ &= E\left[X^2-2m_X X+m_X^2\right] \\ &= E\left[X^2\right]-m_X^2\end{aligned}$$

- σ_X is called the *standard deviation*
  - It is the average distance from the mean, a measure of the concentration of X around the mean

Gaussian Distribution

- Gaussian or normal distribution is a continuous r.v. with pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left[-\frac{1}{2\sigma_X^2}(x - m_X)^2\right]$$



- A Gaussian r.v. is completely determined by its mean and variance, and hence usually denoted as

$$X \sim N\left(m_X, \sigma_X^2\right)$$

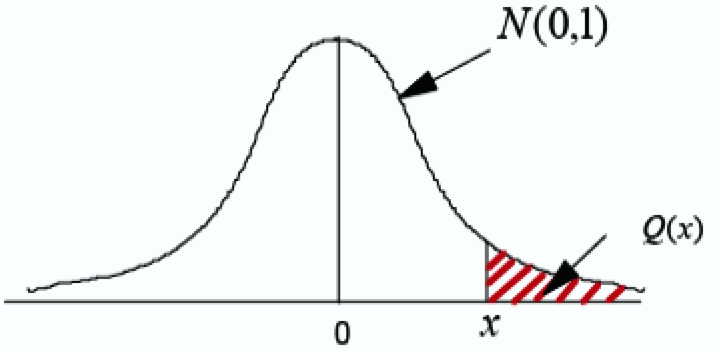
- By far the most important distribution in communications

Q-Function

- The Q-function is a standard form to express error probabilities without a closed form

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du$$

- The Q-function is the area under the tail of a Gaussian pdf with mean 0 and variance 1



- Extremely important in error probability analysis!!!

DIGITAL COMUNICATION SYSTEM
Elements of Statistics
Q-Function

- Q-function is monotonically decreasing
- Some features

$$Q(-\infty) = 1 \qquad Q(0) = \frac{1}{2} \qquad Q(\infty) = 0 \qquad Q(-x) = 1 - Q(x)$$

- Upper bound

$$Q(x) \quad \leq \frac{1}{2} e^{-x^2/2}$$

- If we have a Gaussian variable $X \sim N(\mu, \sigma^2)$, then

$$\Pr(X > x) = Q\left(\frac{x - \mu}{\sigma}\right)$$

Joint Distributions

- Consider 2 r.v.'s X and Y, joint distribution function is defined as

$$F_{XY}(x, y) = P(X \le x, Y \le y)$$

and joint PDF is $$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y}$$

- Key properties of joint distribution

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{XY}(x, y)dxdy = 1$$

$$P(x_1 < X \le x_2, y_1 < Y \le y_2) = \int_{y_1}^{y_2}\int_{x_1}^{x_2} p_{XY}(x, y)dxdy$$

Joint Distributions

- Marginal distribution

$$P_X(x) = P(X \le x, -\infty < Y < \infty) = \int_{-\infty}^{\infty}\int_{-\infty}^{x} p_{XY}(\alpha,\beta)\,d\alpha\,d\beta$$

$$P_Y(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} p_{XY}(\alpha,\beta)\,d\alpha\,d\beta$$

- Marginal density

$$p_X(x) = \int_{-\infty}^{\infty} p_{XY}(x,\beta)\,d\beta$$

- X and Y are said to be *independent* iff

$$P_{XY}(x,y) = P_X(x)P_Y(y)$$

$$p_{XY}(x,y) = p_X(x)p_Y(y)$$

Correlation

- Correlation of the two r.v. X and Y is defined as

$$R_{XY} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{XY}(x,y)dxdy$$

- Correlation of the two centered r.v. X-E[X] and Y-E[Y], is called the covariance of X and Y

$$\sigma_{XY} = E\Big[\big(X - E[X]\big)\big(Y - E[Y]\big)\Big]$$

$$= E[XY] - E[X]E[Y]$$

- If $\sigma_{XY} = 0$, i.e. $E[XY] = E[X]E[Y]$, then X and Y are called uncorrelated.

Correlation

- The covariance of X and Y normalized w.r.t. $\sigma_X \sigma_Y$ is referred to the correlation coefficient of X and Y:

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

- If X and Y are independent, then they are uncorrelated.
- The converse is not true (except the Gaussian case)

Joint Gaussian Random Variables

- $X_1, X_2, \ldots, X_n$ are jointly Gaussian iff

$$f(x_1, x_2, \ldots, x_n) = \frac{1}{(2\pi)^{n/2}(\det(\mathbf{C}))^{1/2}} \exp\left[-\frac{(\mathbf{x}-\mathbf{m})^T\mathbf{C}^{-1}(\mathbf{x}-\mathbf{m})}{2}\right]$$

  - x is a column vector $\mathbf{x} = (x_1, \ldots, x_n)^T$
  - **m** is the vector of the means $\mathbf{m} = (m_1, \ldots, m_n)^T$
  - **C** is the $n \times n$ covariance matrix

$$\mathbf{C} = \left[C_{i,j}\right] \qquad C_{i,j} = E\left[(X_i - m_i)(X_j - m_j)\right]$$

Elements of Statistics

Two variate Gaussian PDF

- Given two r.v.s: X_1 and X_2 that are joint Gaussian

$$\mathbf{C} = \begin{bmatrix} E[(X_1 - m_1)^2] & E[(X_1 - m_1)(X_2 - m_2)] \\ E[(X_1 - m_1)(X_2 - m_2)] & E[(X_2 - m_2)^2] \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

- Then

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[\frac{(x_1 - m_1)^2}{\sigma_1^2} - \frac{2\rho(x_1 - m_1)(x_2 - m_2)}{\sigma_1\sigma_2} + \frac{(x_2 - m_2)^2}{\sigma_2^2} \right] \right\}$$

- For uncorrelated X and Y, i.e. $\rho = 0$

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left\{-\frac{1}{2}\left[\frac{(x_1 - m_1)^2}{\sigma_1^2} + \frac{(x_2 - m_2)^2}{\sigma_2^2}\right]\right\}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-(x_1 - m_1)^2/2\sigma_1^2} \cdot \frac{1}{\sqrt{2\pi}\sigma_2} e^{-(x_2 - m_2)^2/2\sigma_2^2}$$

$$= f(x_1) f(x_2)$$

⇨ X1 and X2 are also independent

If X_1 and X_2 are Gaussian and uncorrelated, then they are independent.

Some Properties of Jointly Gaussian r.v.s

- If n random variables $(X_1, X_2, \cdots, X_n)$ are jointly Gaussian, any set of them is also jointly Gaussian. In particular, all individual r.v.s are Gaussian
- Jointly Gaussian r.v.s are completely characterized by the mean vector and the covariance matrix, i.e. the second-order properties
- Any linear combination of $(X_1, X_2, \cdots, X_n)$ is a Gaussian r.v.

Law of Large Numbers

- Consider a sequence of r.v. $\{X_1, X_2, \ldots, X_n\}$
- Let $$Y = \frac{1}{n}\sum_{i=1}^{n} X_i$$
- If X_i's are uncorrelated with the same mean m_X and variance $\sigma_X^2 < \infty$
- Then $$\lim_{n\to\infty} P\left(|Y - m_X| \geq \varepsilon\right) = 0 \qquad \forall \varepsilon > 0$$

$\boxed{\text{the average converges to the expected value}}$

Central Limit Theorem

- If $\{X_1, X_2, \ldots, X_n\}$ are i.i.d random variables with common mean m_X and common variance σ_X^2
- Then, $Y = \frac{1}{n}\sum_{i=1}^{n} X_i$ converges to a $\mathcal{N}\left(m_X, \frac{\sigma_X^2}{n}\right)$

> the sum of many i.i.d random variables converges to a Gaussian random variable

- Thermal noise results from the random movement of many electrons – it is well modeled by a Gaussian distribution.