Digital Communications
Transmission: Basics on Modulation

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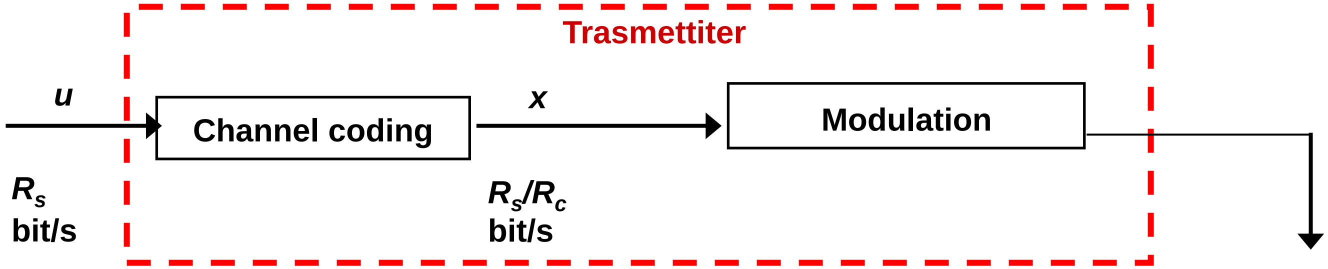
DIGITAL COMMUNICATION SYSTEM
Transmission Systems



Transmitter

The transmitter has the objective to transform the input signal (which is digital in a DCS) in a signal that is adapted to be transmitted over a specific physical medium (fiber, cable, wireless).
This "adaptation" is performed by the **MODULATION**.
However, the transmitter also includes other blocks to make the transmission over the channel, such as the **channel coding**
The transmitter performs a mapping from the message set A to the signal set S

with
$$S = \{s_0, s_1, \ldots\ldots, s_{m-1}\}$$

s_i are in general complex number REPRESENTING real waveforms

DIGITAL COMUNICATION SYSTEM
Transmission Systems

Topics:

1) Meaning of modulation
2) Baseband vs passband transmission
3) Superheterodyne receiver
4) Quick overview of analog modulation and in particulat, amplitude modulation
5) Complex representation of passband signals

DIGITAL COMMUNICATION SYSTEM
Modulation

Why this "adaptation is needed"?

1) In case the input signal is a digital signal (a sequence of digital symbols), we need to trasform them into a "continous time" waveform
2) Both in case of digital or analog input signal, the waveform must be compatible with the characteristics of the channel (e.g. bandwidth)

Baseband modulation: uses basis functions (waveforms) that have most of their energy at low frequencies. Waveforms usually take the form of shaped pulses (baseband channels are: twisted-pairs, coaxial cables etc.)

Bandpass modulation: use basis functions that have energy centered at or near a carrier or center frequency: shaped pulses modulate a sinusoid called a carrier wave or simply a carrier; for radio transmission the carrier is converted to an electromagnetic field for propagation to the desired destination (bandpass channels: radio-frequency wireless channel)

Why we need to use a carrier for the radio transmission of a baseband channel and hence, use the bandpass modulation?

DIGITAL COMUNICATION SYSTEM
Modulation

Bandpass modulation

Bandpass modulation (either analog or digital) is the process by which an information signal is converted to a sinusoidal waveform;

For digital modulation, such a sinusoid of duration T is referred to as a digital symbol.

The sinusoid has just three features that can be used to distinguish it from other sinusoids: amplitude, frequency and phase.

The bandpass modulation can be defined as the process whereby the amplitude, frequency or phase on an RF carrier, or a combination of them, is varied in accordance with the information to be transmitted.

Bandpass modulation

Let us consider a carrier wave, or just a carrier:

$$s_0(t) = A\cos\theta(t) = A\cos(2\pi f_0 t + \varphi) = A\cos(\omega_0 t + \varphi)$$

ω_0 radian frequency
φ phase

Let us denote with *m(t)* the modulating signal (i.e., the information signal)

The modulated signal (i.e. the signal transmitted over the channel) is:

$$s(t) = \Theta(m(t), s_0(t))$$

where $\Theta(\cdot)$ is a transformation (which must be revertible to be able to demodulate the signal)

Bandpass demodulation

When the receiver exploits knowledge of the carrier's phase to detect signals the process is called **coherent detection**
When the receiver does not utilize such phase reference information, the process is called **noncoherent detection**

Note: in digital communication the terms demodulation and detection are often used interchangeably
but
Demodulation emphasizes waveforms recovery
Detection the process of symbol decision

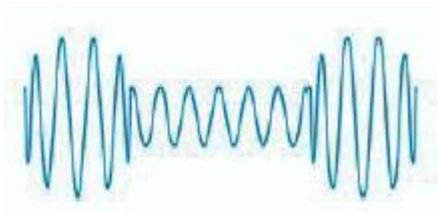
In ideal coherent detection, there is available at the receiver a prototype of each possible arriving signal.
These prototype waveforms attempt to duplicate the transmitted signal set in every respect, even RF phase.
The receiver is then said *phase locked* to the incoming signal.
In case of noncoherent detection, the receiver is simplier as it does not need to perform phase estimation but the price paid is an increased probability of error.

DIGITAL COMMUNICATION SYSTEM
Elements of analog modulation

In case of analog modulation, the modulating signal is analog.

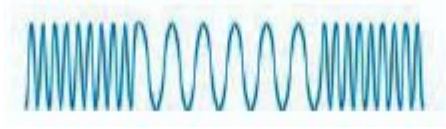
Amplitude modulation

$$s(t) = (A + K_A m(t))\cos(\omega_0 t + \phi_0)$$



Frequency modulation

$$s(t) = A\cos(\omega_0 t + \phi_0 + 2\pi K_F \int_{-\infty}^{t} m(\tau)d\tau)$$



Phase modulation

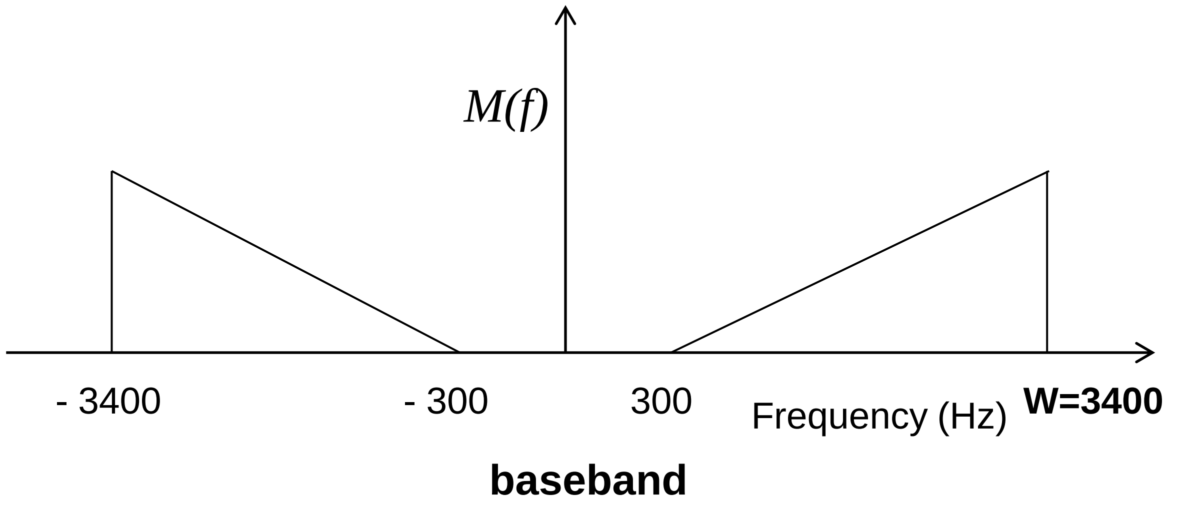
$$s(t) = A\cos(\omega_0 t + \phi_0 + K_\phi m(t))$$

where K_A, K_F , K_ϕ are constantds which determinates “how deep” is the modulation.

DIGITAL COMMUNICATION SYSTEM
Elements of analog modulation

Amplitude modulation

Modulating signal



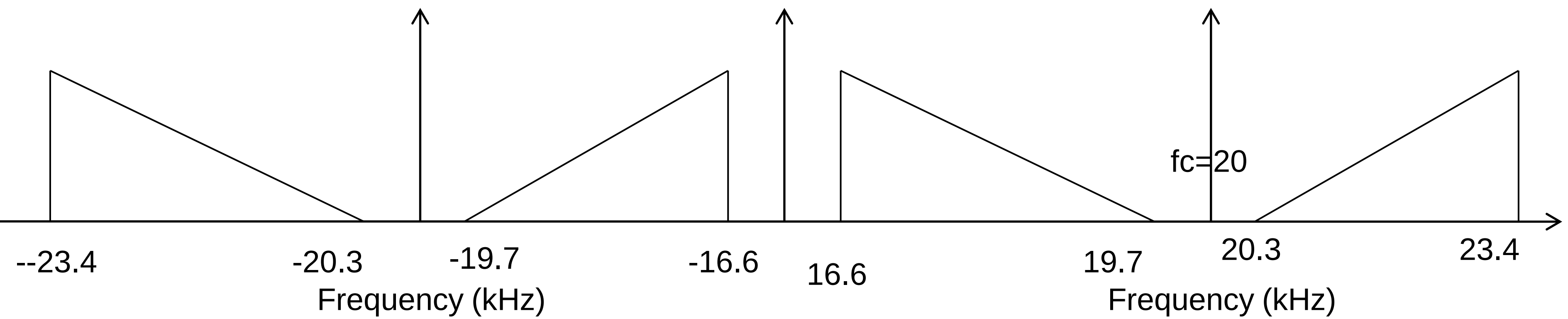
Bandwidth: W

Amplitude modulation

Double-Side Band (DSB)

$$s(t) = (A + K_A m(t))\cos(\omega_0 t + \phi_0)$$

$$S(f) = \frac{1}{2}A(\delta(f - f_0) + \delta(f + f_0)) + \frac{K_A}{2}\left(M(f - f_0) + M(f + f_0)\right)$$

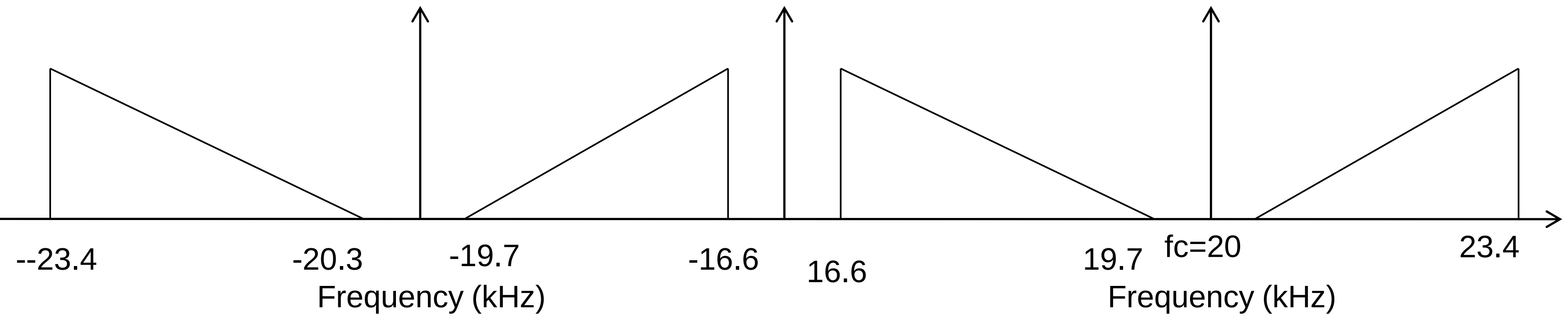


Double sideband

DIGITAL COMMUNICATION SYSTEM
Elements of analog modulation

Amplitude modulation

Double-Side Band (DSB)



Required channel bandwidth: $2W$

Required transmit power: $\frac{A^2}{2} + \frac{K_A^2}{2} P_M$

P_M → power of the modulating signal

AM is wasteful of bandwidth

AM is wasteful of power

DIGITAL COMMUNICATION SYSTEM
Elements of analog modulation
Linear modulation

In its simplest form, the linear modulation is expressed by the following relation:

$$v(t) = s(t)\cos(\omega_0 t + \phi_0)$$

Note: it is related to the amplitude modulation but with respect to it, it does not contain the carrier. Moreover, the amplitude modulation itself is not linear.

•DSB-SC is a linear modulation

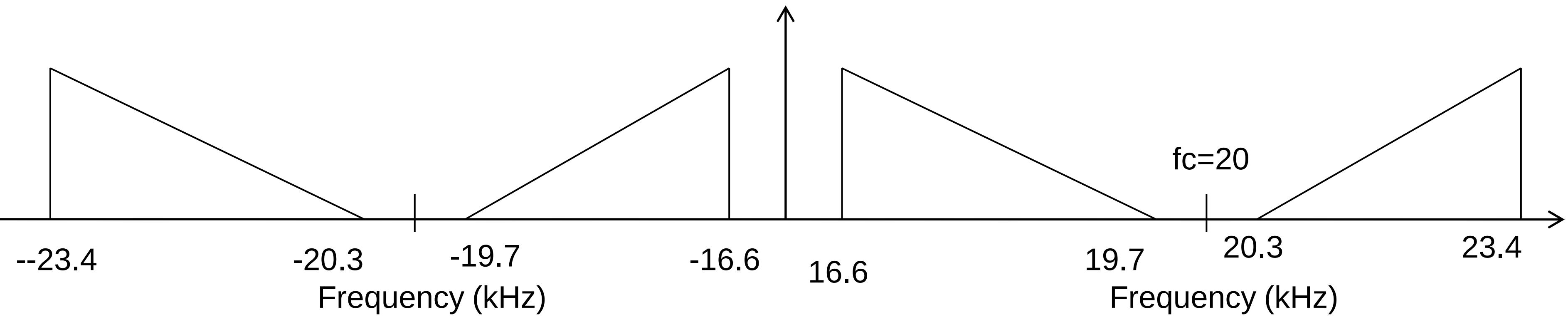
Other types of linear modulations are :
•SSB (single sideband)
•VSB (vestigial sideband)

In the rest of the course we will mainly deal with linear modulators that are needed to place the bandwidth of the digital transmitted signal within the bandwidth of the transmission channel ⇨ performing the so called **frequency translation**

Linear modulation

DSB-SC (DSB-Suppressed Carrier)

$$S(f) = \frac{1}{2} A(\delta(f - f_0) + \delta(f + f_0)) + \frac{K_A}{2}(M(f - f_0) + M(f + f_0))$$



Double sideband suppressed carrier (DSBSC)

Required channel bandwidth: $2W$

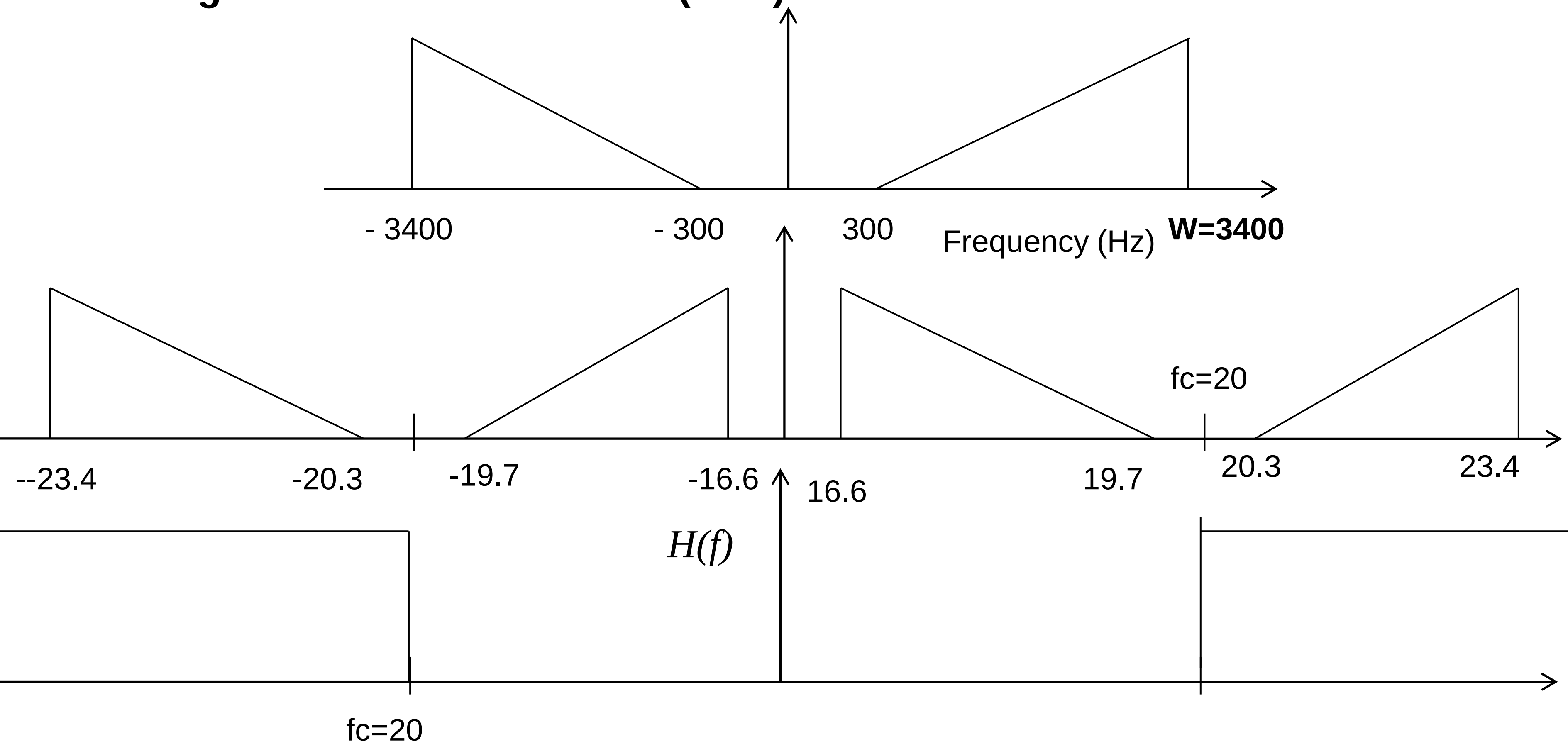
Required transmit power: $\frac{K_A^2}{2} P_M$

power of the modulating signal

DSB-SC is wasteful of bandwidth

DIGITAL COMMUNICATION SYSTEM
Linear modulation

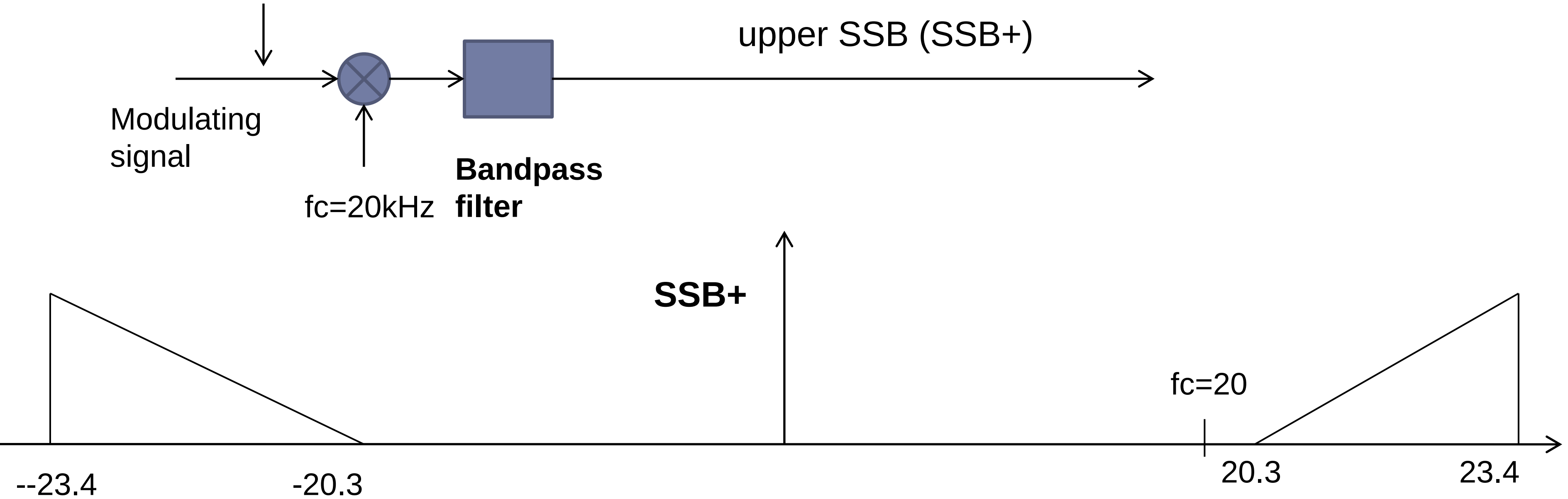
Single-sideband Modulation (SSB)



H(f) is a bandpass filter

DIGITAL COMMUNICATION SYSTEM
Linear modulation

Single-sideband Modulation (SSB)



Very efficient in bandwidth but…….(let us analyze more carefully by using the complex representation)

DIGITAL COMMUNICATION SYSTEM
Complex Representation

Very important

In the frequency domain bandpass signals are characterized by a symmetrical spectral extension (symmetrical with respect to the origin) and a bandwidth that is much smaller than the central frequency

In the time domain, they have an oscillating behaviour similar to that of a sinusoidal waveforms, with frequency equal to the central frequency and amplitude slowly varying (examples are amplitude modulated signals).

To work with these signals, instead of using the traditional signal theory, it is more efficient to use the so-called complex representation in which a real signal $v(t)$ is represented as the real part of a complex signal called **analytic signal:**

$$v(t) = \mathrm{Re}\left[z(t)\right]$$

↓

analytic signal

DIGITAL COMUNICATION SYSTEM
Complex Representation
Analytic signal

The analytic signal of a real signal v(t) si defined as:

$$v(t) = \mathrm{Re}\left[z(t)\right] \qquad \text{condition (1)}$$

However, to be a “representation” of v(t) it must uniquely identify v(t) and this is not the case if we only use the condition (1).
According to (1):

$$z = v + j\hat{v}$$

where the in quadrature component $\hat{v}$ is undefined by now
The condition that completely defines z is that

<u>the spectrum of z has only positive frequencies</u> condition (2)

DIGITAL COMUNICATION SYSTEM
Complex Representation
Analytic signal

$$v(t) = \underbrace{\frac{1}{2} z(t)}_{v_+} + \underbrace{\frac{1}{2} z(t)^*}_{v_-}$$

v_+ is the part of the spectrum at positive frequency

v_- is the part of the spectrum at negative frequency

⇓

The operation of real part gives to the original signal the negative spectrum which was removed by the operation of generating the analytic signal.

DIGITAL COMUNICATION SYSTEM
Complex Representation
Analytic signal

Let be *v(t)* a generic real signal. Its analytic signal is the output of the following filter:

$$H_z(f) = 2 \cdot 1(f) = \begin{cases} 2 & \mathrm{f} > 0 \\ 0 & \mathrm{f} < 0 \end{cases}$$

$$z = h_z * v = Cx[v]$$

$Cx[\cdot]$ Is the operator “take the complex part of”

$$h_z(t) = \delta(t) + j\frac{1}{\pi t}$$

Proof of condition (1):

$$\mathrm{Re}[z] = \mathrm{Re}[h_z] * v = \delta(t) * v = v$$

DIGITAL COMUNICATION SYSTEM
Complex Representation
Quadrature Component or Hilbert Transform

Once the analytic signal has been achieved, it is possible to calculate the in-quadrature component of *v(t)*:

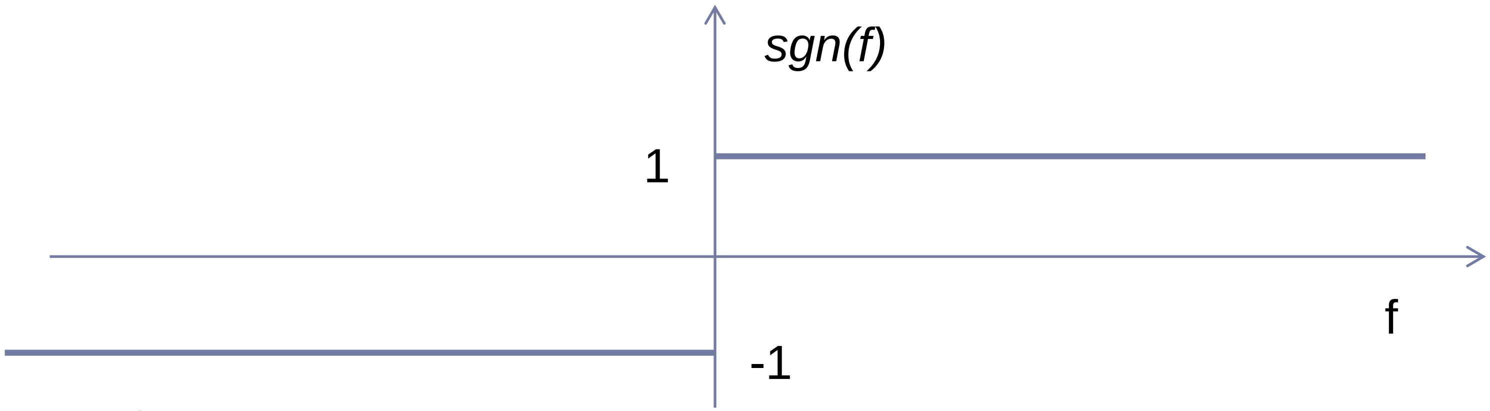
$$\hat{v} = \mathrm{Im}[z] = \mathrm{Im}[h_z * v] = h_q * v$$

$$h_q(t) = \mathrm{Im}[h_z] = \frac{1}{\pi t}$$

$$\Downarrow$$

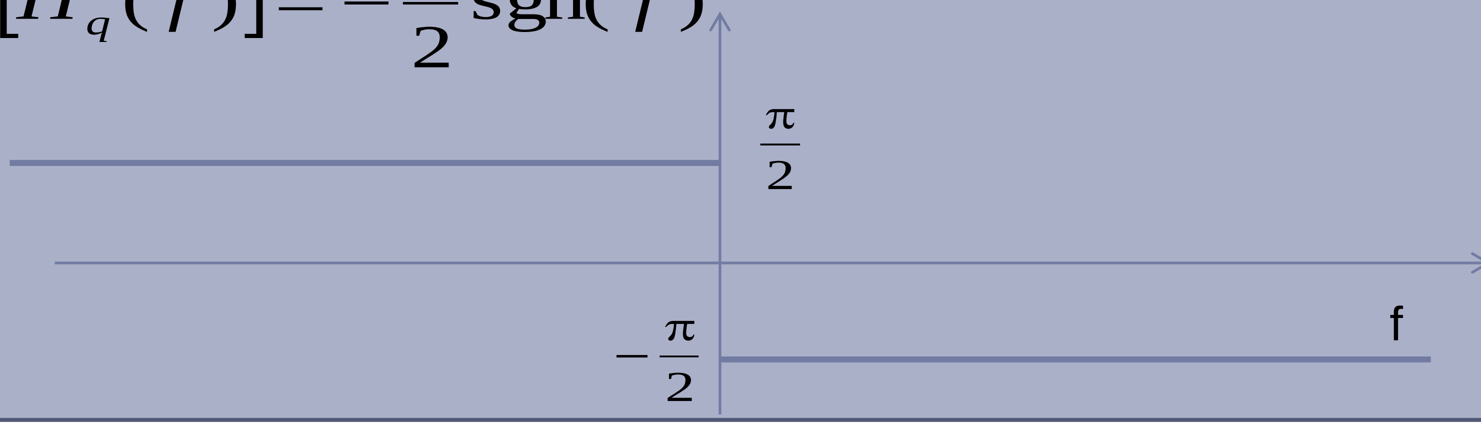
$$H_q(f) = \frac{1}{2j}\left[H_z(f) - H_z^*(-f)\right] = -j\,\mathrm{sgn}(f)$$

DIGITAL COMUNICATION SYSTEM
Complex Representation
Quadrature Component or Hilbert Transform



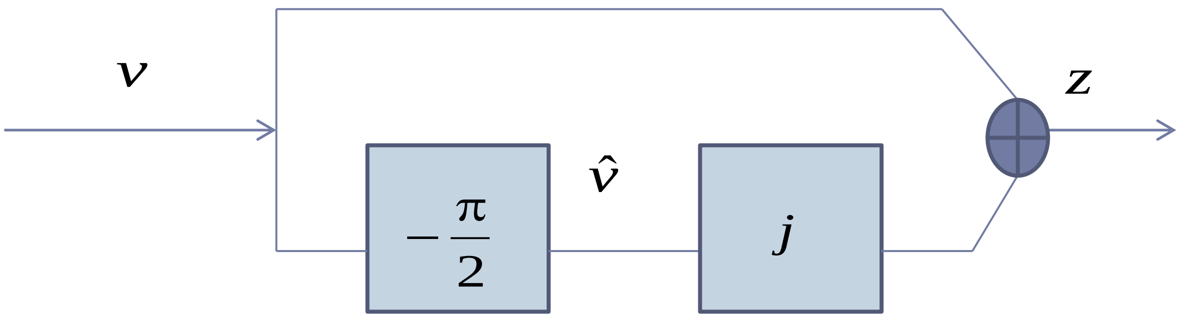
$$\left|H_q(f)\right| = 1 \quad \forall \mathrm{f}$$

$$\arg\left[H_q(f)\right] = -\frac{\pi}{2}\,\mathrm{sgn}(f)$$



Ideal phase shifter

DIGITAL COMUNICATION SYSTEM
Complex Representation
Quadrature Component or Hilbert Transform



The part of signal v(t) at positive frequency are phase shifted of $-\frac{\pi}{2}+\frac{\pi}{2}=0$

They sum coherently

The part of signal v(t) at negative frequency are phase shifted of $\frac{\pi}{2}+\frac{\pi}{2}=\pi$

They sum in phase opposition

This explains why the amplitude of z is double at positive frequency and zero at negative frequency

Complex Representation

Quadrature Component or Hilbert Transform

The quadrature component $\hat{v}$ is also called Hilbert Transform of v:

$$\hat{v} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v(\tau)}{t-\tau} d\tau$$

Homework - prove that (hint: use the concept of phase shifter)

$$v = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{v}(\tau)}{t-\tau} d\tau$$

DIGITAL COMUNICATION SYSTEM
Complex Representation
Complex envelope

It is also convenient to translate the positive frequency mode around the origin so to achieve a BASEBAND signal which is a representation of the original bandpass signal and it is called **COMPLEX ENVELOPE**
The complex envelope is the baseband representation of bandpass signals

Let us choose a reference frequency f_0

The complex envelope of v(t) with respect to f_0 is defined as:

$$c(t) = z(t)e^{-j2\pi f_0 t}$$

DIGITAL COMUNICATION SYSTEM
Complex Representation
Complex envelope

The spectral extension of the complex envelope is

$$\mathscr{E}(c(t)) = \mathscr{E}(z(t)) - f_0$$

if $$\mathscr{E}(z(t)) = (f_0 - B_1),\ f_0 + B_2)$$

$$\mathscr{E}(c(t)) = (-B_1, B_2)$$

If the reference frequency belongs to the positive part of the spectrum of the original signal, then the complex envelope is at baseband and it is called the **baseband equivalent** of the considered signal

Note: the reference frequency could be arbitrary but it is usually chosen in a convenient way, for instance, in case of bandpass signals with sinusoidal carrier the natural choice is the carrier frequency.

DIGITAL COMUNICATION SYSTEM
Complex Representation
Baseband Components

Very important

$$a(t) = \mathrm{Re}[c(t)]$$ In-phase baseband component of v(t)

$$b(t) = \mathrm{Im}[c(t)]$$ In-quadrature baseband component of v(t)

$$c(t) = z(t)e^{-j2\pi f_0 t} = (v(t) + j\hat{v}(t))(\cos 2\pi f_0 t - j\sin 2\pi f_0 t)$$

⇓

$$a(t) = v(t)\cos 2\pi f_0 t + \hat{v}(t)\sin 2\pi f_0 t$$

$$b(t) = \hat{v}(t)\cos 2\pi f_0 t - v(t)\sin 2\pi f_0 t$$

⇓

DIGITAL COMUNICATION SYSTEM
Complex Representation
Baseband Components

Very important

$$v(t) = \mathrm{Re}\left[(a(t) + jb(t))(\cos 2\pi f_0 t + j\sin 2\pi f_0 t)\right]$$

⇓

$$v(t) = a(t)\cos 2\pi f_0 t - b(t)\sin 2\pi f_0 t \qquad (*)$$

⇓

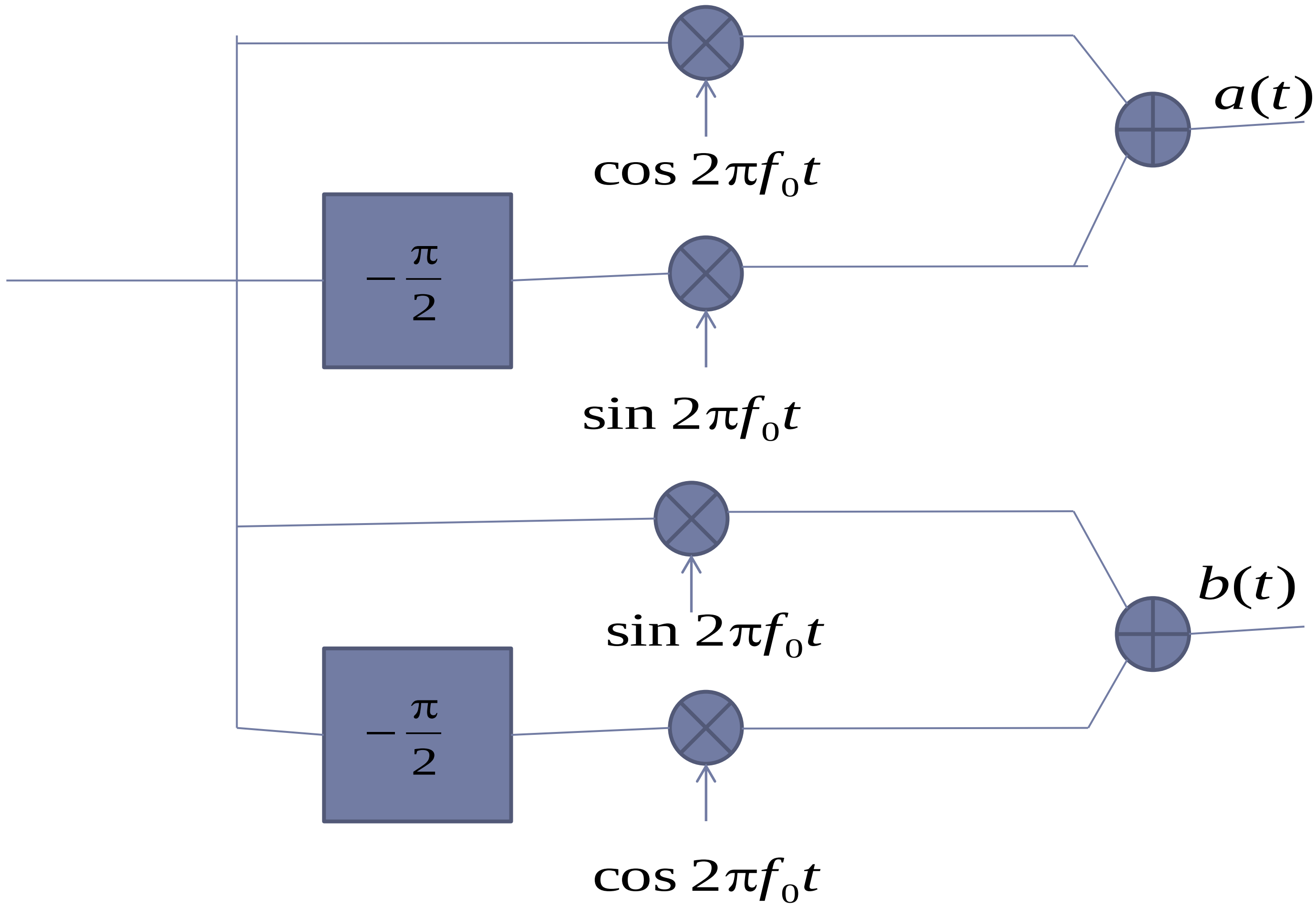
Any real bandpass signal can be written as in(*) where a(t) and b(t) are baseband signals and hence, slowly varying signals.

DIGITAL COMUNICATION SYSTEM

Complex Representation

Baseband Components

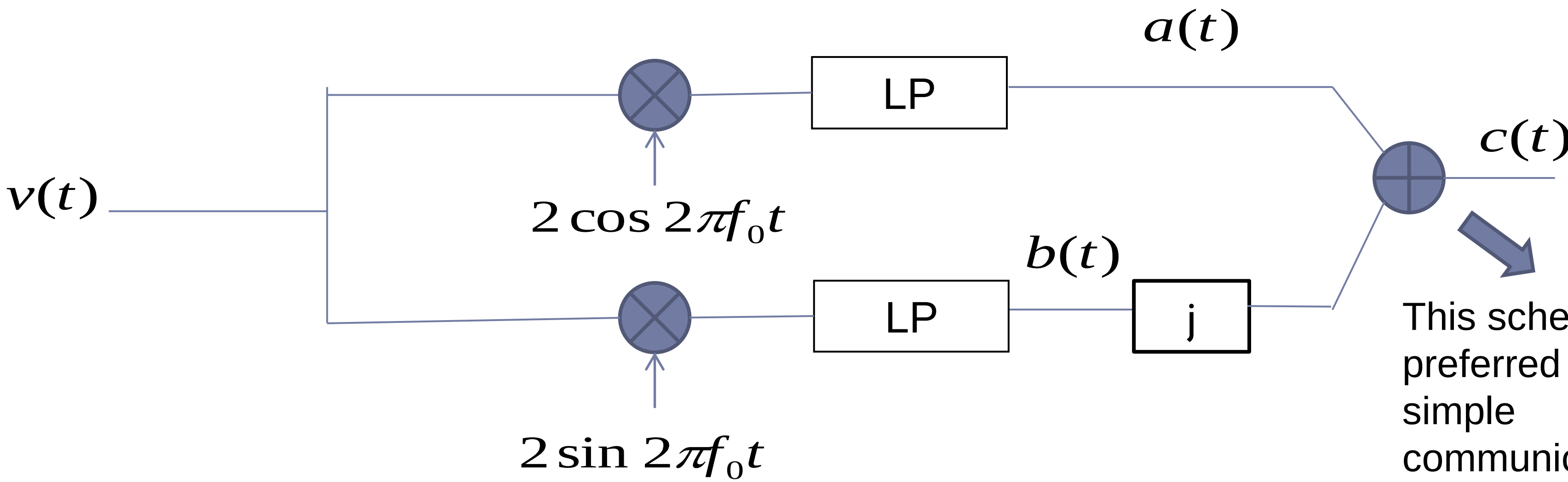
Very important



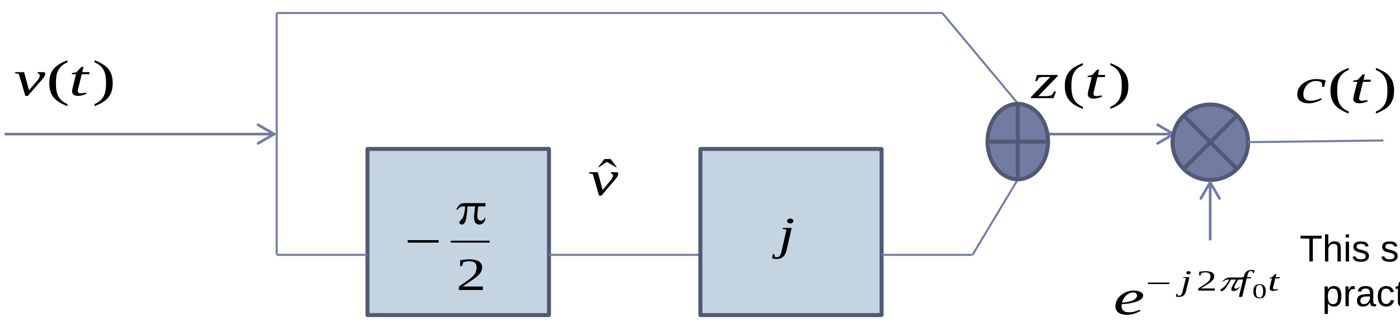
DIGITAL COMUNICATION SYSTEM

Complex Representation

Complex demodulators for the generation of the baseband equivalent



This scheme is preferred in simple communication system



This scheme have some practical performance advantages for the implementation of carrier-phase recovery systems

Homework:

Find the complex envelope of a bandpass filter

$$G(f) = rect\left(\frac{f - f_c}{2B}\right) + rect\left(\frac{f + f_c}{2B}\right)$$

DIGITAL COMUNICATION SYSTEM
Complex Representation

Product of two signals
Let us assume

$$v(t) = a(t)q(t)$$

baseband with spectral extension (-B,B)

passband with spectral extension $(f_q, +\infty)$ with $f_q > 0$

$$q = q_+ + q_- \quad \text{with} \quad q_- = q_+^*$$

$$v(t) = a(t)q_+(t) + a(t)q_+^*(t)$$

Spectral extension $(-B + f_q, +\infty)$

DIGITAL COMUNICATION SYSTEM
Complex Representation

Product of two signals

$$\text{if} \quad f_q > B$$

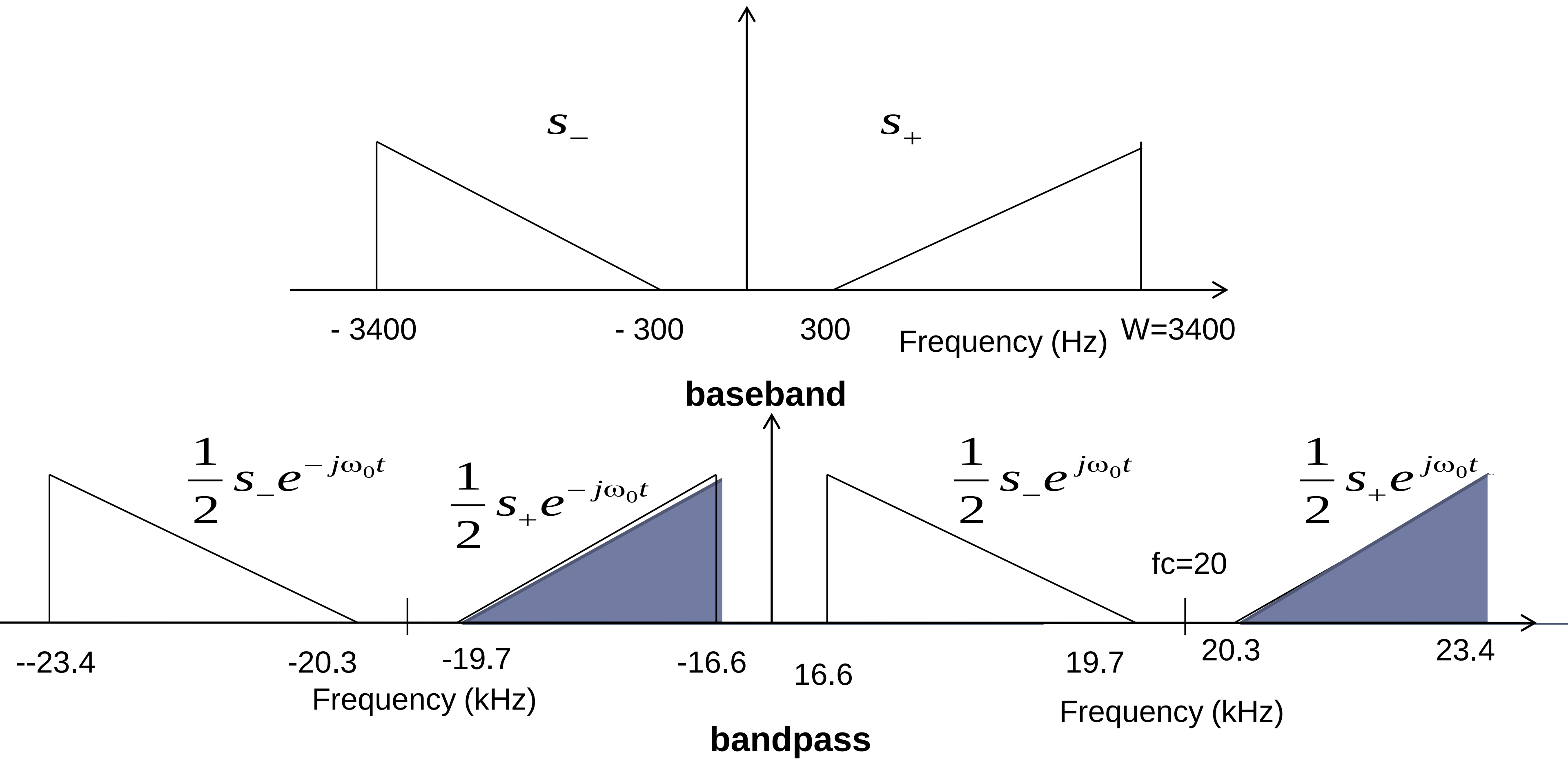
$$v_+(t) = a(t)q_+(t)$$

⇓

$$z(t) = 2v_+(t) = 2a(t)q_+(t) = a(t)z_q(t)$$

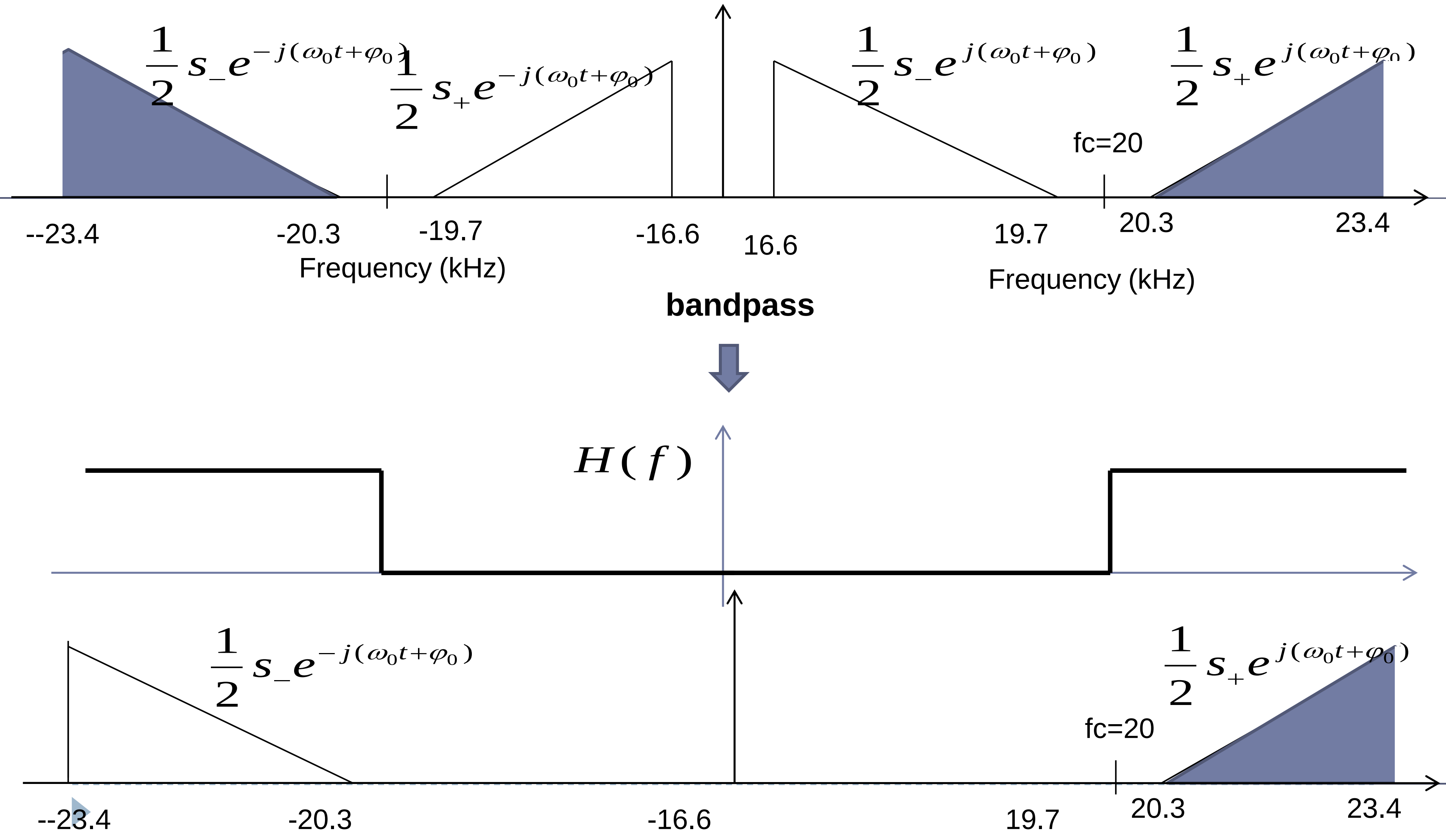
DIGITAL COMUNICATION SYSTEM
Complex Representation

Double side band modulation



DIGITAL COMUNICATION SYSTEM
Complex Representation

single side band modulation: upper sideband



DIGITAL COMUNICATION SYSTEM
Complex Representation

single side band modulation: upper sideband

$$v(t) = \frac{1}{2} s_- e^{-j(\omega_0 t + \phi_0)} + \frac{1}{2} s_+ e^{j(\omega_0 t + \phi_0)} = \mathrm{Re}\left[s_+(t) e^{j(\omega_0 t + \phi_0)} \right]$$

$$\boxed{s_-(t) = s_+^*(t)}$$

Demodulator?

As exercise, prove that the demodulator is the same as the one of the DSB-SC scheme, i.e. a product for a replica of the carrier an then a lowpass filter.

BUT

<u>It can be also shown that an error in the carrier phase recovery is much more critical in case of SSB than in case of DSB</u>

DIGITAL COMUNICATION SYSTEM
Complex Representation

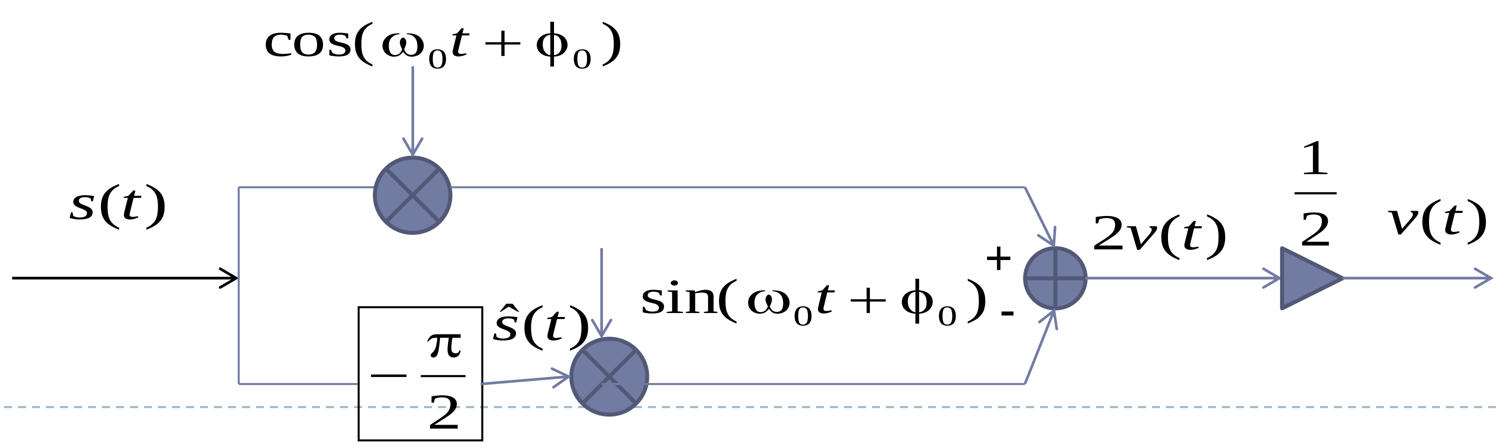
single side band modulation: upper sideband

Alternative scheme for the upper SSB Modulator

$$s_+(t) = \frac{1}{2}\left[s(t) + j\hat{s}(t)\right]$$

$$v(t) = \frac{1}{2}\mathrm{Re}\left[(s(t) + j\hat{s}(t))\cos(\omega_0 t + \phi_0) + j\sin(\omega_0 t + \phi_0)\right] =$$

$$= \frac{1}{2}\left[s(t)\cos(\omega_0 t + \phi_0) - \hat{s}(t)\sin(\omega_0 t + \phi_0)\right]$$



single side band modulation: upper sideband

Homework:
find the alternative scheme for the modulator for the lower SSB

single side band modulation: upper sideband

Comparisons of the two modulators schemes

(let us refer to them as scheme #1 and scheme #2, where the #2 is the alternative one)

- The scheme #2 makes the filtering (ideal phase shifter) at low frequency (which is easier from the HW implementation point of view)
- Both schemes MUST rely on one IDEAL component:
  - The scheme #1 relies on an <u>ideal</u> low pass filter
  - The scheme #2 on an ideal phase shifter

Let us assume that the spectral extension in baseband is $(f_{S,} F_S)$

In both cases, it is more difficult to approximately meet these requirements of ideality when the ratio between

$$f_S \,/\, F_S$$

is small (in other terms, the more the signal s(t) has a high spectral content around the origin)

VSB

For the DSB, the ratio f_s / F_s is not critical for the filtering at the demodulator as it is possible to implement it easily also when $f_s = 0$

On the other hand, it has a double bandwidth wrt the SSB

In case of SSB, it is difficult to well separate the upper from the lower sideband when f_s / F_s is small.
The VSB is a trade-off between the other two. The filter that partially remove the lower sideband has the following frequency response:

$$H(f) = 0 \qquad f_0 - F_s < f < f_0 - \alpha F_s$$
$$0 < H(f) < 1 \qquad f_0 - \alpha F_s < f < f_0 + \alpha F_s$$
$$H(f) = 1 \qquad f_0 + \alpha F_s < f < f_0 + F_s$$

If in the part between the passband and the suppressed band, a symmetry is verified with respect the carrier $H(f_0) = \frac{1}{2}$

Th demodulation is possible with the same scheme as DBS and SSB and the problem of the filtering is relaxed.

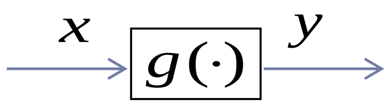
DIGITAL COMUNICATION SYSTEM
Spectral extension of some signals

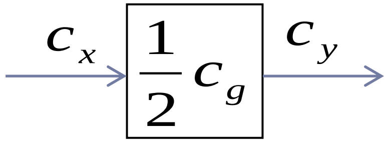
Type of signal	Lower cutoff frequency	Upper cutoff frequency	Nominal bandwidth
telephone	300Hz	3400Hz	4kHz
AM radio	100Hz	5000Hz	5kHz
FM radio	50Hz	15000Hz	15kHz
White and black TV	50Hz	5MHz	5MHz
Colour TV	50Hz	5MHz	5MHz

Only in case of telephone, the radio $f_S / F_S \approx 10$ allows to implement the SSB modulation without much difficulty.

DIGITAL COMUNICATION SYSTEM
Complex Representation
Baseband Equivalent of passband filters

Let us consider a generic REAL PASSBAND system (real means that to any real input corresponds a real output), not necessarily linear, where both input *x(t)* and output *y(t)* are passband.
Let us consider a frequency f_0 and represent input and output signals with their complex envelope c_x c_y
Let us find the system that has as input c_x and output c_y as the *complex equivalent in passband* of the original system.





Theorem
A real filter with impulse response g(t) and frequency response G(f) has as baseband equivalent a filter with an impulse response $\frac{1}{2}c_g$ *which is half of the complex envelope of g(t) and its frequency response is:*

$$\frac{1}{2}C_g(f) = \frac{1}{2}Z_g(f + f_0) = G_+(f + f_0)$$

DIGITAL COMUNICATION SYSTEM
Complex Representation
Baseband Equivalente of passband filters

$$y = g * x = (g_+ + g_-) * (x_+ + x_-) =$$
$$= g_+ * x_+ + g_- * x_- + g_+ * x_- + g_- * x_+$$

- $g_+ * x_+$: Contains only positive frequencies
- $g_- * x_-$: Contains only negative frequencies
- $g_+ * x_- + g_- * x_+$: They are equal to zero as convolution of signal with not overlapping bandwidths

$$y_+ = g_+ * x_+ \qquad y_- = g_- * x_-$$

The original filter can be divided in two filters, one acting on the positive and one on the negative frequencies. Moreover, the second filter is redundant as it involves only signals that are complex conjugated of the ones of the first filter.

<u>We can study only the first filter who gives the relation among the analytic signals</u>

DIGITAL COMUNICATION SYSTEM
Complex Representation
Baseband Equivalent of passband filters

$$y_+ = g_+ * x_+$$

$$z_y = \frac{1}{2} z_g * z_x, \quad \frac{1}{2} z_g = g_+$$

$$c_y(t) = z_y(t) e^{-j\omega_0 t} = \int_{-\infty}^{\infty} \frac{1}{2} z_g(t-\tau) e^{-j\omega_0 (t-\tau)} z_x(\tau) e^{-j\omega_0 \tau} d\tau$$

$$c_y(t) = \int_{-\infty}^{\infty} \frac{1}{2} c_g(t-\tau) c_x(\tau) d\tau = \frac{1}{2} c_g(t) * c_x(t)$$

DIGITAL COMUNICATION SYSTEM
Complex Representation
Baseband Equivalent of passband filters

Filtering of the baseband components

$$c_y(t) = \frac{1}{2} c_g(t) * c_x(t)$$

$$a_y(t) = \frac{1}{2} a_g(t) * a_x(t) - \frac{1}{2} b_g(t) * b_x(t)$$
$$b_y(t) = \frac{1}{2} b_g(t) * a_x(t) + \frac{1}{2} a_g(t) * b_x(t)$$

It is one filter with two real inputs (e.g., the baseband component of the signal in input x) and two real outputs (e.g., the baseband components of y)
The matrix of the impulse response is:

$$\begin{bmatrix} \frac{1}{2} a_g(t) & -\frac{1}{2} b_g(t) \\ \frac{1}{2} b_g(t) & \frac{1}{2} a_g(t) \end{bmatrix}$$

DIGITAL COMUNICATION SYSTEM

Complex Representation

Baseband Equivalent of passband filters

<u>Filtering of the baseband components</u>

The frequency responses of the two filters are:

$$\frac{1}{2}A_g(f)=\frac{1}{4}\left[C_g(f)+C_g^*(-f)\right]=\frac{1}{2}\left[G_+(f+f_0)+G_f^*(-f+f_0)\right]$$

$$\frac{1}{2}B_g(f)=\frac{1}{4j}\left[C_g(f)-C_g^*(-f)\right]=\frac{1}{2j}\left[G_+(f+f_0)-G_f^*(-f+f_0)\right]$$

One condition that is often required is the the filtering is diagonal, e.g.:

$$a_y(t)=\frac{1}{2}a_g(t)*a_x(t)$$

$$b_y(t)=\frac{1}{2}a_g(t)*b_x(t)$$

In other terms, <u>the two components pass through the system without interacting with each other.</u>

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Baseband Equivalent of passband filters

Filtering of the baseband components

The diagonal condition implies that

$$B_g(f) = \frac{1}{j}\left[G_+(f+f_0) - G_f^*(-f+f_0)\right] = 0$$

$$\Downarrow$$

$$G_+(f+f_0) = G_f^*(-f+f_0)$$

$$\Downarrow$$

Hermitian symmetry with respect to f_0

Homework:
verify whether the ideal bandpass filter corresponds to a diagonal filtering

Superheterodyne receiver

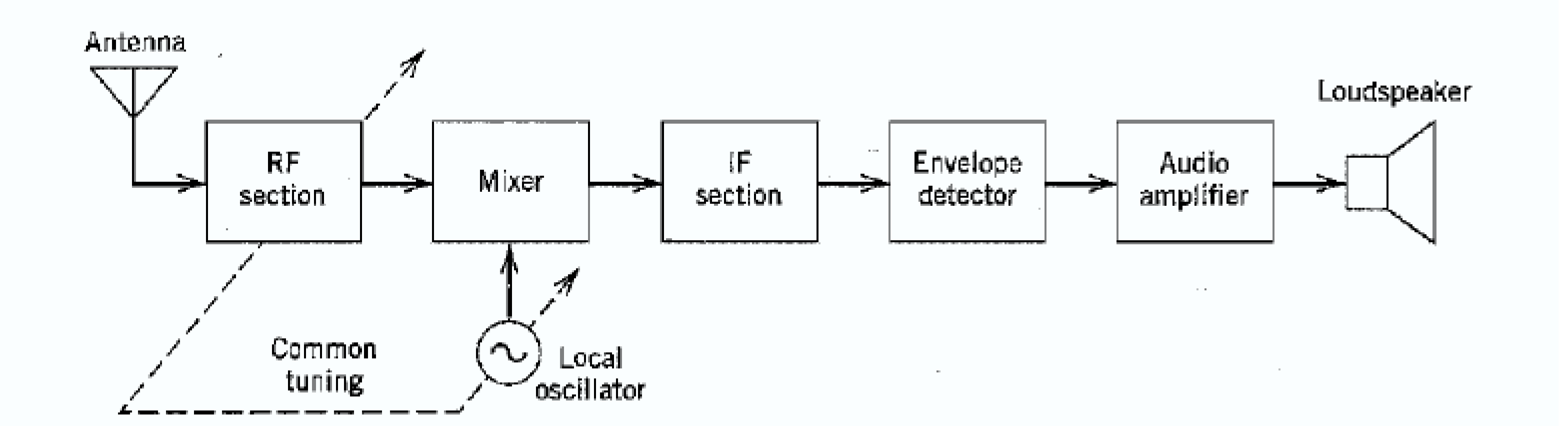
It is type of receiver that allows to perform some type of functions, which a typical receiver must perform, in an elegant and practical way.

These functions are:

- ❑Carrier-frequency tuning
- ❑Filtering
- ❑amplification

All radio and TV receivers now being made are superheterodyne receivers.

It consists of: a radio-frequency (RF) section, a mizer and a local oscillator, an intermediate frequency (IF) section, demodulator and power amplifier.



Superheterodyne receiver

Typical frequency parameters of AM and FM radio receivers

	AM Radio	*FM Radio*
RF carrier range	0.535–1.605 MHz	88–108 MHz
Midband frequency of IF section	0.455 MHz	10.7 MHz
IF bandwidth	10 kHz	200 kHz

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Spectral analysis

Direct problem: given the spectrum of the passband signal v(t), find the spectrum of the complex envelope

Inverse problem: given the spectrum of the compex envelope , find the spectrum of the passband signal

Three cases of interest:

1) Deterministic signals (this case is rather straighforward and will not be treated in the following)
2) WSS random processes (e.g., noise)
3) Cyclostationary random processes with period equal to the carrier frequency (e.g., modulated signals)

Review of some concepts for random processes: couple of processes

Mutual correlations

$$r_{xy}(t,\tau) \equiv E\left[x(t+\tau)y^*(t)\right]$$
$$r_{yx}(t,\tau) \equiv E\left[y(t+\tau)x^*(t)\right]$$

If two random processes are jointly stationary (in correlation), the two above correlations are independent from the time:

$$r_{xy}(\tau) = r_{yx}^*(-\tau)$$
$$R_{xy}(f) = R_{yx}^*(f)$$

Note: mutual spectral densities are complex in general. On the other hand, auto-correlation are always real.
For real processes, mutual correlation are real and hence, mutual spectral densities have Hermitian symmetry.

Review of some concepts for random processes: couple of processes

Two random processes are ORTHOGONAL if their mutual correlation is identically zero:

$$r_{xy}(\tau) = 0 \qquad \tau \in R$$

And they are uncorrelated if the mutual covariance is zero:

$$r_{xy}(\tau) - m_x m_y^* = 0 \qquad \tau \in R$$

Structural constraints

$$0 \le \left| r_{xy}(\tau) \right|^2 \le r_x(0) r_y(0)$$

$$0 \le \left| R_{xy}(f) \right|^2 \le R_x(f) R_y(f)$$

⇓

<u>If they have disjoint bandwidths, the two processes are orthogonal</u>

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Review of some concepts for random processes: couple of processes

$$\text{if } R_{xy}(f) = 0 \qquad \forall f$$

Orthogonal processes

$$\text{if } R_{xy}(f) = R_x(f) R_y(f) \qquad \forall f$$

Parallel processes (or perfectly correlated)

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Review of some concepts for random processes: couple of processes

$$x(t) = a(t) + jb(t)$$

For the stationarity of x, it is not sufficient that a and b are stationary -they must be JOINTLY stationary
Moreover, to know the power of a and b it is not sufficient to know the one of x but also of its conjugate.

$$r_{xx^*}(\tau) = r_{xx^*}(-\tau)$$
$$R_{x^*}(f) = R_x^*(-f) = R_x(f)$$
$$R_{x^*x}(f) = R_{xx^*}^*(-f) = R_{xx^*}^*(f)$$

Review of some concepts for random processes: couple of processes

If x(t), a WSS random process, is given in input to a filter with impulse response g(t), the output process y(t) is WSS and also jointly stationary with the input process. The correlations are then:

$$r_{yx}(\tau) = g * r_x(\tau)$$
$$r_y(\tau) = g_-^* * r_{yx}(\tau) = g_-^* * g * r_x(\tau) = c_g * r_x(\tau)$$

$$\Downarrow$$

$$R_{yx}(f) = G(f)R_x(f)$$
$$R_y(f) = G^*(f)R_{yx}(f) = |G(f)|^2 R_x(f)$$

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Review of some concepts for random processes: couple of processes

if y_1 e y_2 Are achieved by filtering the process v(t) with two filters with frequency responses $H_1(f)$ and $H_2(f)$

$$W_{y_1 y_2}(f) = H_1(f) H_2^*(f) W_v(f)$$

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Complex Representation

Spectral analysis

Direct problem

Let use consider a <u>real stationary process</u> v(t) with autocorrelation and spectral density:

$$R_v(\tau) \leftrightarrow W_v(f)$$

What is the spectral density of the part of v(t) at positive frequencies?

$$W_{v_+}(f) = |H_+(f)|^2 W_v(f) = 1(f) W_v(f)$$

$$W_{v_+ v_-}(f) = H_+(f) H_-^*(f) W_v(f) = 0$$

$$W_v(f) = W_{v_+}(f) + W_{v_-}(f) \qquad (*)$$

If v+ and v- were not orthogonal, in the (*) also the mutual spectral densities would appear. Moreover, it is expected as v must have symmetric spectral density as it is real

Spectral analysis

Direct problem

$$W_v(f) = W_{v_+}(f) + W_{v_+}(-f)$$

as $v_- = v_+^*$

$$z = 2v_+ \qquad z^* = 2v_-$$

$$W_z(f) = 4W_{v_+}(f)$$
$$W_{zz^*}(f) = 0$$

$$R_c(\tau) = R_z(\tau)e^{-j\omega_0\tau} \leftrightarrow W_c(f) = W_z(f + f_0) = 4W_{v_+}(f + f_0)$$

The complex envelope is a stationary process.

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Direct problem

$$R_{cc^*}(t,\tau) = E\left[c(t)c(t+\tau)\right] =$$
$$= E\left[z(t+\tau)e^{-j\omega_0(t+\tau)}z(t)e^{-j\omega_0 t}\right] =$$
$$= E\left[z(t+\tau)z(t)\right]e^{-j(2\omega_0 t+\omega_0\tau)} = R_{zz^*}(t,\tau)e^{-j(2\omega_0 t+\omega_0\tau)} = 0$$

⇓

The complex envelope is orthogonal with his coniugate

Statistical powers

$$M_{xy} = E\left[x(t)y^*(t)\right] = \int_{-\infty}^{\infty} W_{xy}(f)df = R_{xy}(0)$$
$$M_{x} = E\left[x(t)x^*(t)\right] = \int_{-\infty}^{\infty} W_{x}(f)df = R_{x}(0)$$

$$M_{v_+} = \int_{-\infty}^{\infty} W_{v_+}(f)df =$$
$$\int_{-\infty}^{\infty} 1(f)W_v(f)df = \frac{1}{2}\int_{-\infty}^{\infty} W_v(f)df = \frac{1}{2}M_v = M_{v_-}$$
$$M_{v_+v_-} = 0$$

$$M_z = 4M_{v_+} = 2M_v, \quad M_{zz^*} = 0$$

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Statistical powers

$$M_c = M_z = 2M_v,\ \ M_{cc^*} = 0$$
$$M_{\hat{v}} = M_v,\ \ M_{v\hat{v}} = 0$$
$$M_a = M_b = M_v,\ \ M_{ab} = 0$$

Homework: Let us consider the stationary process n(t) with power spectral density:

$$W_n(f) = W_0 rect\left(\frac{f - f_c}{2B}\right) + W_0 rect\left(\frac{f + f_c}{2B}\right)$$

Calculate spectral density of z, c, a, b and the related statistical powers.

Inverse problem

Let us consider the complex envelope $c(t)$ of a random process

Hypotheses:

1) $c(t)$ is a stationary random process

2) $c(t)$ and $c^*(t)$ are orthogonal

from

$$R_c(\tau) = R_z(\tau)e^{-j\omega_0\tau} \leftrightarrow W_c(f) = W_z(f + f_0) = 4W_{v_+}(f + f_0)$$

$$R_z(\tau) = R_c(\tau)e^{j\omega_0\tau} \leftrightarrow W_z(f) = W_c(f - f_0)$$

z(t) is stationary

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Inverse problem

Let us consider the complex envelope $c(t)$ of a random process

Hypotheses:

1) $c(t)$ is a stationary random process

2) $c(t) \text{ and } c^*(t)$ are orthogonal

$$R_{zz^*}(t,\tau) = R_{cc^*}(t,\tau)e^{j\omega_0\tau + j2\omega_0 t} = 0$$

Due to the hypothesis 2)

⇓

$z(t) \text{ and } z^*(t)$ are orthogonal

$$W_z(f) = W_c(f - f_0), \qquad W_{zz^*}(f) = 0$$

DIGITAL COMUNICATION SYSTEM
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Inverse problem

By considering that $$v(t) = \frac{1}{2}\left[z(t) + z^*(t)\right]$$

$$R_v(\tau) = E\left[v(t+\tau)v(t)\right] =$$

$$= \frac{1}{4}E\left[z(t+\tau)z(t) + z^*(t+\tau)z^*(t) + z(t+\tau)z^*(t) + z^*(t+\tau)z(t)\right]$$

$$= \frac{1}{4}\left[R_z(\tau) + R_{z^*}(\tau)\right]$$

$$W_v(f) = \frac{1}{2}\left[W_c(f - f_0) + W_c(-f - f_0)\right]$$

Spectral analysis

<u>Example:</u> modulated signals

the complex envelope can be written as $c(t) = c_0(t)e^{j\phi_0}$
where $c_0(t)$ is a stationary random process and jointly stationary with $c_0^*(t)$
The verification of hypothesis 2) depends from the phase

Case a): ϕ_0 *is* a uniform variable between $(0,2\pi)$

Case b): ϕ_0 a constant

Prove that in case a) the hypothesis 2) is verified whilst in case b), the hypothesis 2) is not verified and the mutual correlation is not zero and the process v(t) is not stationary by cyclostationary.

Complex Representation

Spectral analysis

homework #1

Given the complex envelope $c(t) = V_0 e^{j\phi_0}$

Find the spectral density of the passband signal of which c(t) is the complex envelope in both hypotheses a) and b) of the previous example

homework #2

Consider a generic linear amplitude modulated signal

$$v(t) = a(t)\cos(\omega_0 t + \phi_0)$$

Where a(t) is stationary and the phase of the carrier uniformly distributed random variable. Moreover, the bandwidth of a(t) is lower than the carrier frequency.

Find the spectral density of v(t)