Digital Communications
Optimal Receivers

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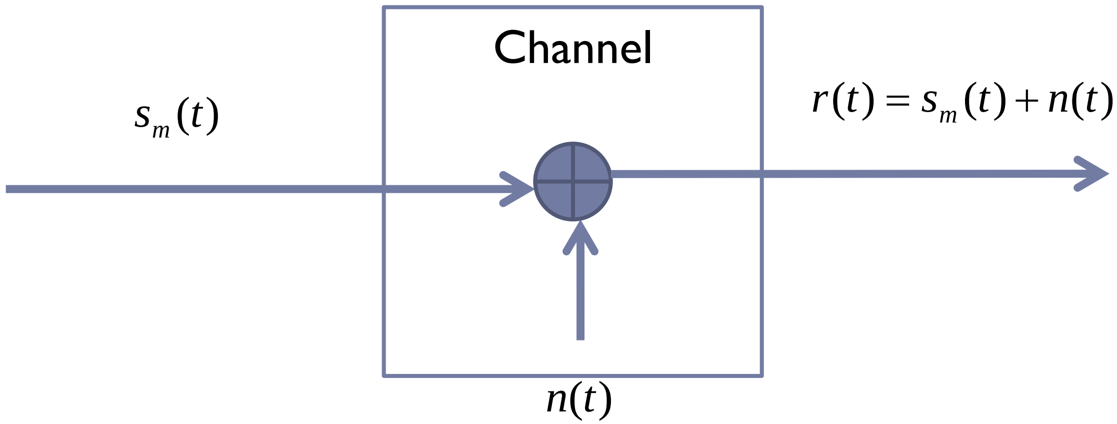
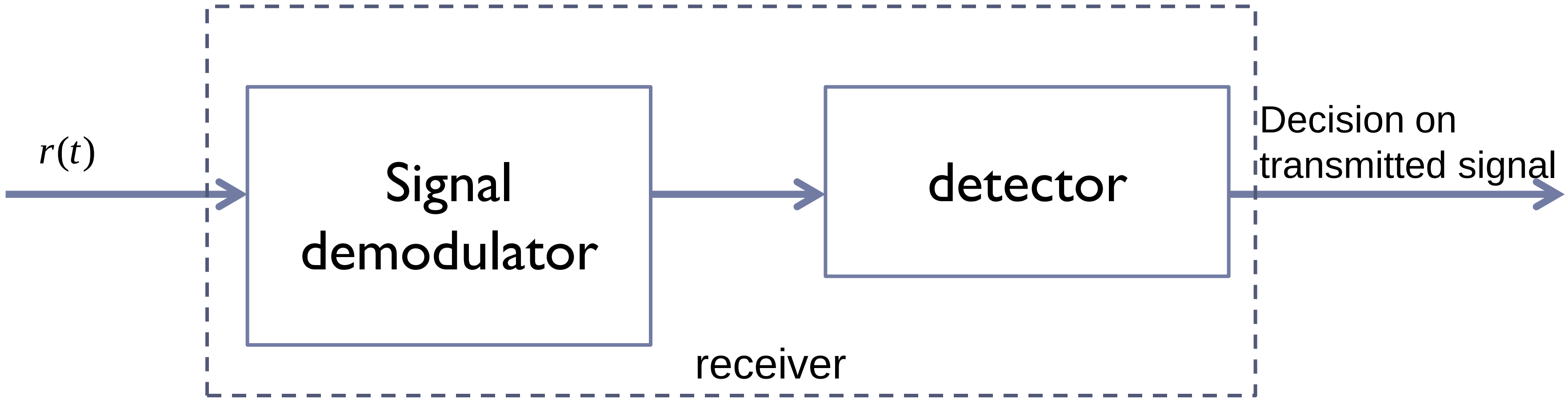
Optimum receiver for binary modulated signals in AWGN



Let us assume that the channel only adds white gaussian noise with power spectral density $N_0 / 2$

How to process the received signal *r(t)* in the interval $0 \le t \le T_b$

Optimum receiver for binary modulated signals in AWGN



It is convenient to subdivide the receiver into two parts

Demodulator: to convert the received signal waveform into a vector $\boldsymbol{y}$ whose dimension is equal to the dimension of the transmitted signal waveforms

Detector: to decide which of the two (in case of binary transmission) possible signal waveforms was transmitted, based on observation of the vector $\boldsymbol{y}$

Optimum receiver for binary modulated signals in AWGN

Correlator-type receiver

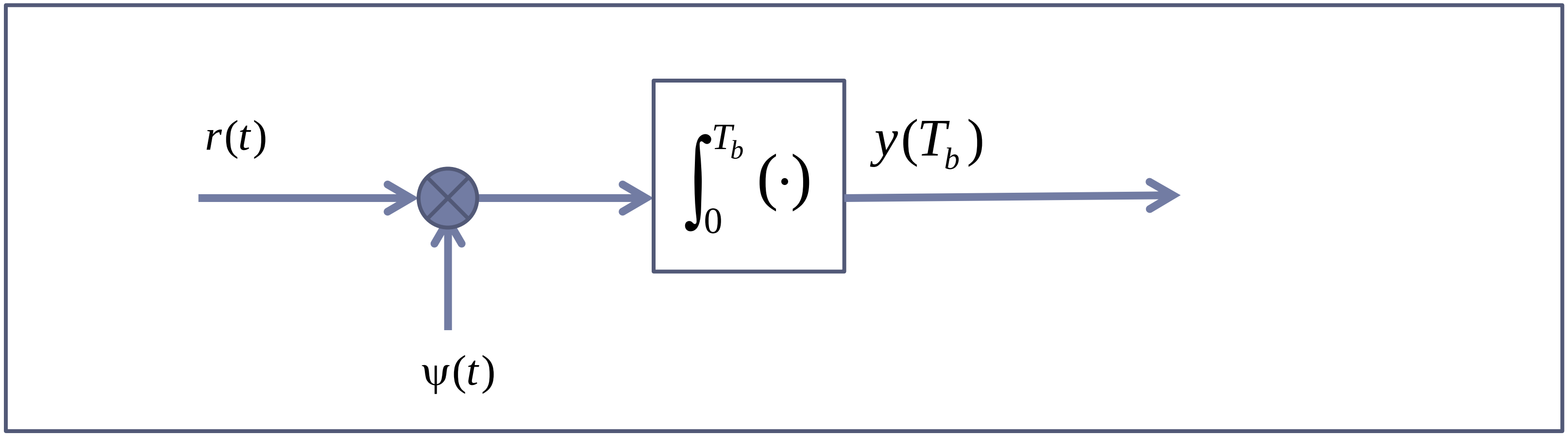
Case #1: binary antipodal signals

$$r(t) = s_m \psi(t) + n(t) \qquad 0 \le t \le T_b$$

where:

$\psi(t)$ is the unit energy rectangular pulse

$s_1 = \sqrt{E_b} \quad s_2 = -\sqrt{E_b}$



Cross-correlator for binary antipodal

Optimum receiver for binary modulated signals in AWGN

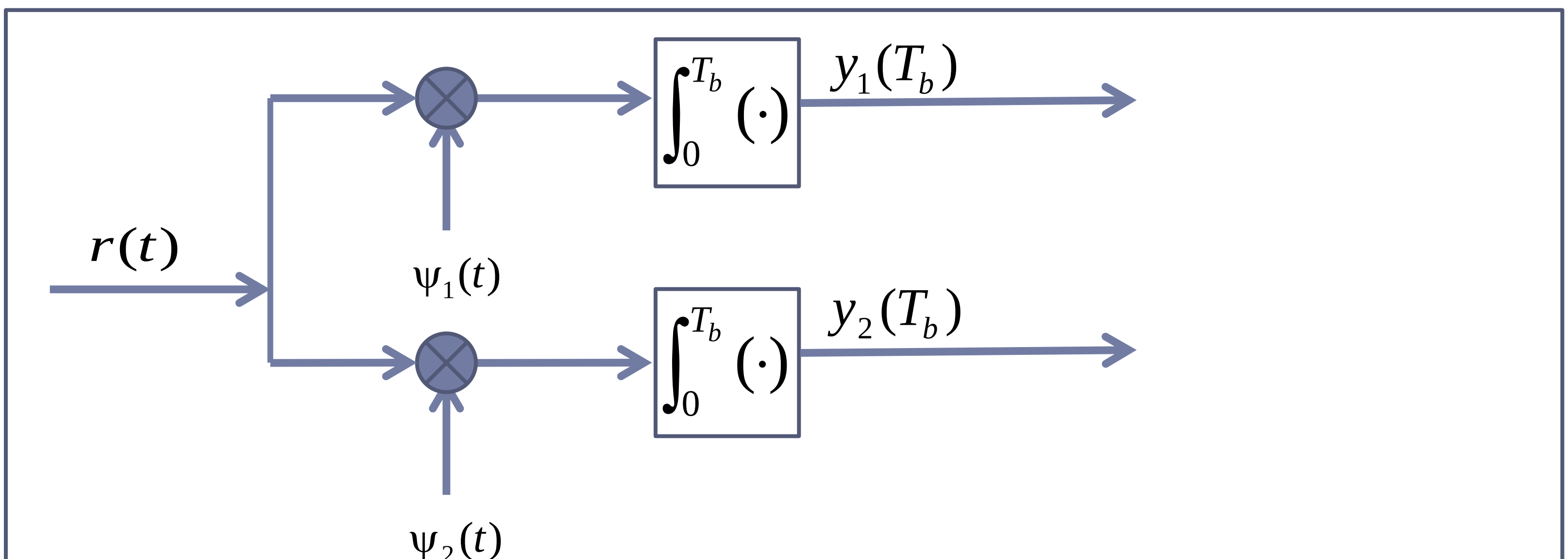
Correlator-type receiver

Case #2: binary orthogonal signals

$$r(t) = s_m(t) + n(t) \qquad 0 \le t \le T_b$$

Where $s_m(t)$ is one of the two following orthogonal waveforms:

$$s_1(t) = s_{11}\psi_1(\mathrm{t}) + s_{12}\psi_2(\mathrm{t})$$
$$s_2(t) = s_{21}\psi_1(\mathrm{t}) + s_{22}\psi_2(\mathrm{t})$$



Cross-correlator for binary orthogonal signals

Optimal receiver in AWGN channel
Matched Filter-type receiver

Fundamental result in communication theory on the:

Detection of a pulse signal of know waveforms that is immersed in <u>additive white noise</u>

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MATCHED FILTER

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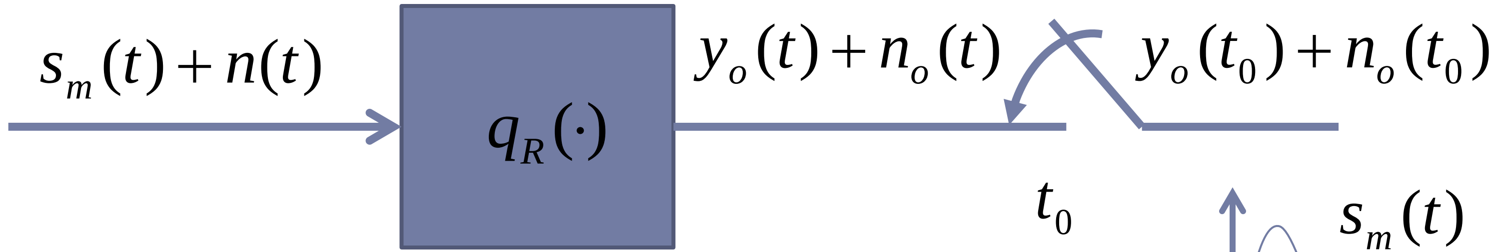
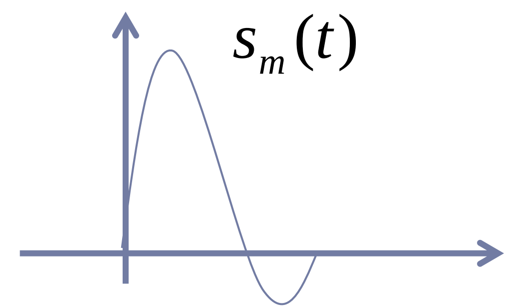
<u>The matched filter (MF) is the optimal linear filter for maximizing the output SNR.</u>

Optimal receiver in AWGN channel
Matched Filter-type receiver





Hypotheses:

1) $s_m(t)$ is a known pulse of finite energy

2) $n(t)$ is a zero-mean noise with spectral density $W_n(f)$

Problem:

Find $q_R(\cdot)$ which maximize the ratio $\dfrac{\left|y_o(t_0)\right|^2}{E\left[\left|n_o(t_0)\right|^2\right]}$

Optimal receiver in AWGN channel
Matched Filter-type receiver

Preliminar considerations:

$$y_o(t) = s_m * q_R(t)$$

$$n_o(t) = n * q_R(t)$$

parseval

$$y_o(t_0) = \int_{-\infty}^{\infty} q_R(\tau) s_m(t_0 - \tau) d\tau = \int_{-\infty}^{\infty} Q_R(f) S_m(f) e^{j2\pi f t_0} df$$

$$E\left[\left|n_o(t_0)\right|^2\right] = \int_{-\infty}^{\infty} W_n(f) \left|Q_R(f)\right|^2 df =$$

Schwartz's inequality

$$\left|\int_{-\infty}^{\infty} X^*(f) Y(f) df\right|^2 \leq \int_{-\infty}^{\infty} \left|X(f)\right|^2 df \int_{-\infty}^{\infty} \left|Y(f)\right|^2 df$$

Equality holds only if $X(f) = kY(f)$

Optimal receiver in AWGN channel
Matched Filter-type receiver

Solution:

$$\frac{\left|y_o(t_0)\right|^2}{E\left[\left|n_o(t_0)\right|^2\right]} = \frac{\left|\int_{-\infty}^{\infty} Q_R(f) S_m(f) e^{j2\pi f t_0}\right|^2}{\int_{-\infty}^{\infty} W_n(f)\left|Q_R(f)\right|^2 df}$$

$$= \frac{\left|\int_{-\infty}^{\infty} Q_R(f)\sqrt{W_n(f)}\,\frac{S_m(f)}{\sqrt{W_n(f)}} e^{j2\pi f t_0} df\right|^2}{\int_{-\infty}^{\infty} W_n(f)\left|Q_R(f)\right|^2 df}$$

$$\leq \int_{-\infty}^{\infty}\left|\frac{S_m(f)}{\sqrt{W_n(f)}} e^{j2\pi f t_0}\right|^2 df \leq \int_{-\infty}^{\infty} \frac{\left|S_m(f)\right|^2}{W_n(f)} df$$

This is the maximum value

Optimal receiver in AWGN channel
Matched Filter-type receiver

Solution:
The maximum value is achieved when the equality holds, e.g. when:

$$Q_R(f)\sqrt{W_n(f)} = k\left(\frac{S_m(f)}{\sqrt{W_n(f)}}e^{j2\pi f t_0}\right)^*$$

$$\Downarrow$$

$$Q_R(f) = k\,\frac{S_m^*(f)}{W_n(f)}e^{-j2\pi f t_0}$$

$$\Downarrow$$

$$Y_o(f) = k\,\frac{\left|S_m(f)\right|^2}{W_n(f)}e^{-j2\pi f t_0}$$

Optimal receiver in AWGN channel
Matched Filter-type receiver

If the noise is **WHITE noise**

$$W_n(f) = N_0 / 2$$

$$Q_R(f) = kS_m^*(f)e^{-j2\pi f t_0} \quad \longleftrightarrow \quad q_R(t) = ks_m^*(t_0 - t)$$

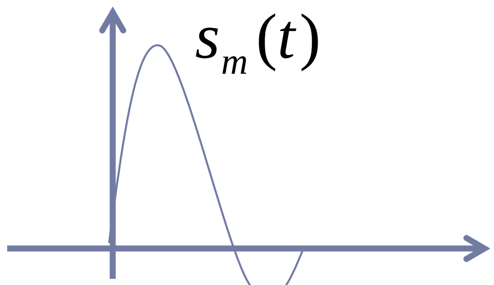
$$Y_o(f) = k\left|S_m(f)\right|^2 e^{-j2\pi f t_0} \quad \longleftrightarrow \quad y_o(t) = kR_{s_m}(t - t_0)$$

Optimal receiver in AWGN channel

Matched Filter-type receiver

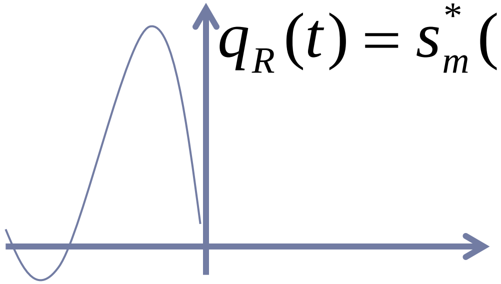
Moreover, if $s_m(t)$ is of finite duration between *(0,T)*

$s_m(t)$

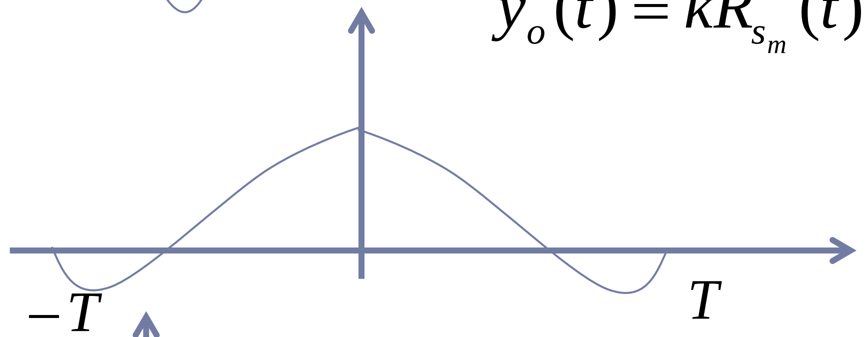


for $t_0 = 0$

$q_R(t) = s_m^*(-t)$

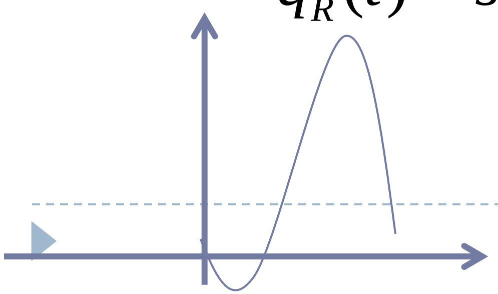


$y_o(t) = kR_{s_m}(t)$

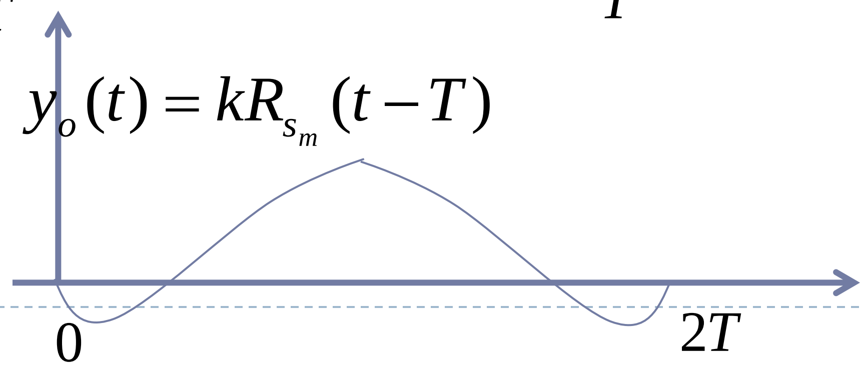


for $t_0 = T$

$q_R(t) = s_m^*(T-t)$



$y_o(t) = kR_{s_m}(t-T)$



Optimal receiver in AWGN channel
Matched Filter-type receiver

Moreover, if $s_m(t)$ is of finite duration between *(0,T)*

$$V_0 = y_o(t_0) = kR_{s_m}(0)$$

$$\sigma^2 = E\left[\left|n_o(t_0)\right|^2\right] = k^2 \frac{N_0}{2} R_{s_m}(0)$$

$$\Downarrow$$

$$\frac{V_0^2}{\sigma^2} = \frac{2E_{s_m}}{N_0}$$

The peak pulse signal-to-noise ratio of a matched filter depend only on the ratio of the signal energy to the power spectral density of the white noise at the filter input and ***NOT on the particular shape of the waveform that is used***

Optimal receiver in AWGN channel
Matched Filter-type receiver

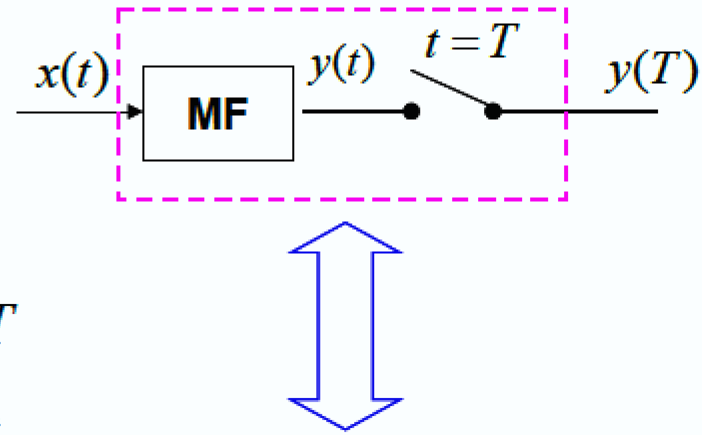
The impulse response of the optimum filter qR, expect for a scaling factor k, is a time-reversed and delayed version of the input signal gR, that is “matched” to the input signal.

No assumption has been made on the statistics of the channel noise, only that is stationary and white.

Optimal receiver in AWGN channel
Matched Filter-type receiver

- Equivalent form – Correlator
  - Let $s_i(t)$ be within $[0, T]$

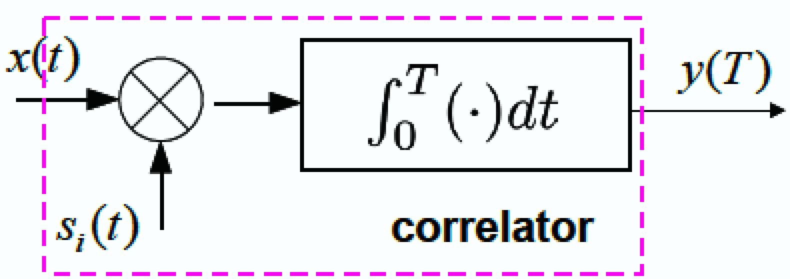
$$y(t) = x(t) * h_m(t) = x(t) * s_i(T-t)$$
$$= \int_0^T x(\tau) s_i(T-t+\tau)\, d\tau$$



  - Observe at sampling time $t = T$

$$y(T) = \int_0^T x(\tau) s_i(\tau)\, d\tau = \int_0^T x(t) s_i(t)\, dt$$

Correlation integration



Optimal receiver in AWGN channel
Matched Filter-type receiver

Examples: MATCHED FILTER of a rectangular pulse

- Consider a rectangular pulse $s(t)$

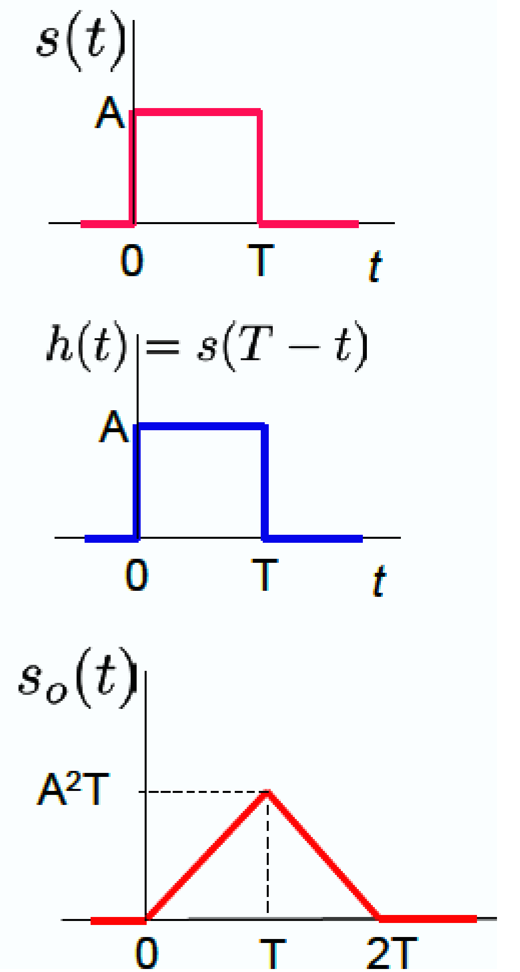
$$E_s = A^2 T$$

- The impulse response of a filter matched to $s(t)$ is also a rectangular pulse

- The output of the matched filter $s_0(t)$ is $h(t) * s(t)$

- The output SNR is

$$(SNR)_o = \frac{2}{N_0}\int_0^T s^2(t)dt = \frac{2A^2T}{N_0}$$

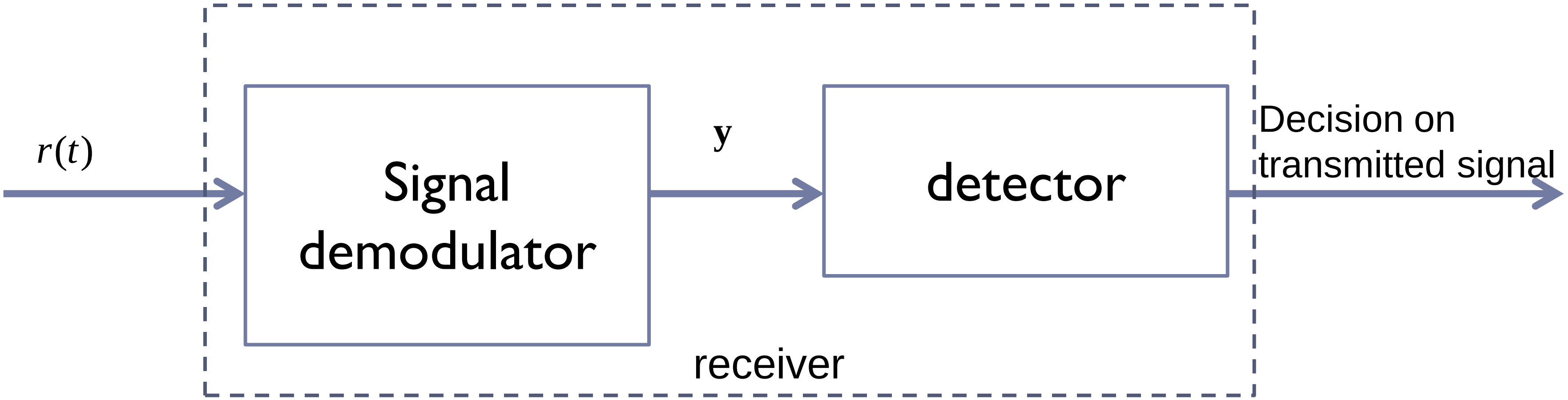


Optimal receiver in AWGN channel

Summary

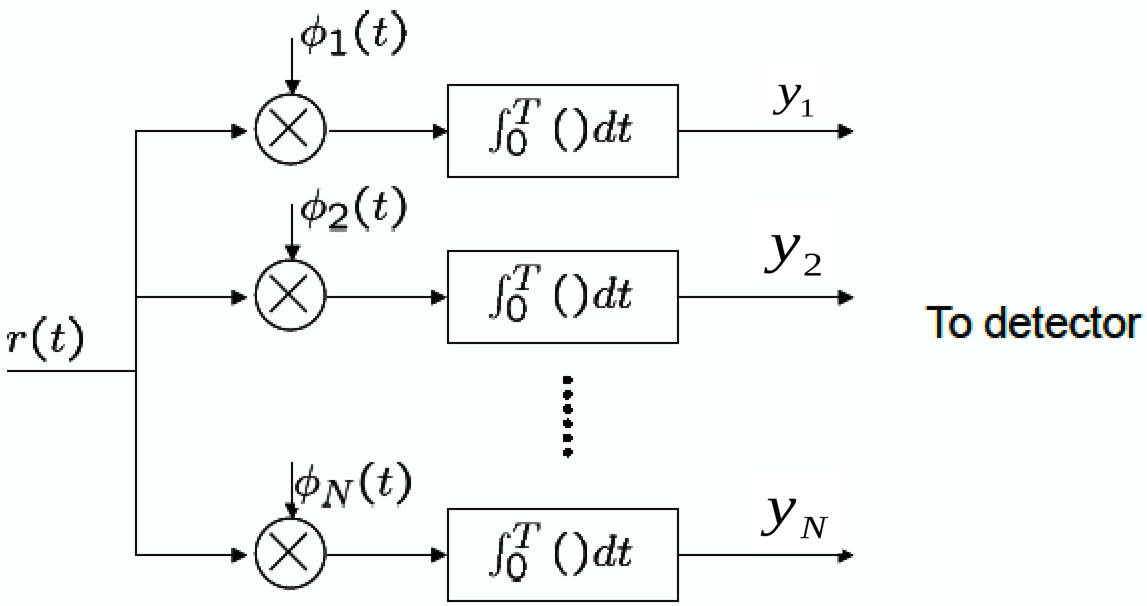
We have demonstrated that for a signal transmitted over an AWGN channel, either a <u>correlation type demodulator</u> or a <u>matched filter type demodulator</u> produces the vector $\boldsymbol{y}$ which contains <u>all the necessary information</u> in $r(t)$

Now, we will discuss the design of a **signal detector** that makes a decision of the transmitted signal in each signal interval based on the observation of $\mathbf{y}$, such that the **probability of making an error is minimized** (or correct probability is maximized)



Optimal receiver in AWGN channel

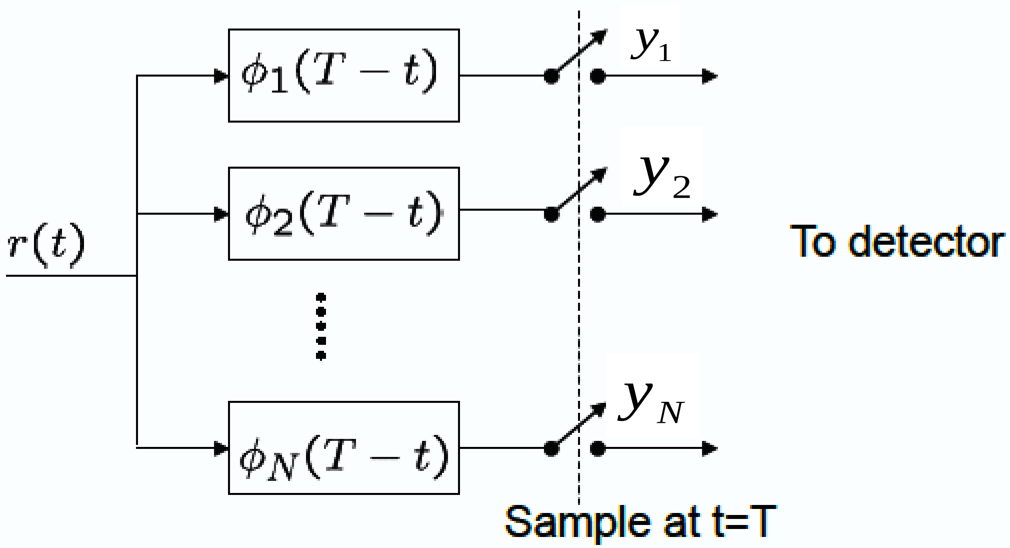
- The received signal r(t) is passed through a parallel bank of N cross correlators which basically compute the projection of r(t) onto the N basis functions $\{\phi_k(t), k = 1, \ldots N\}$



Optimal receiver in AWGN channel

- Alternatively, we may apply the received signal r(t) to a bank of N matched filters and sample the output of filters at t = T. The impulse responses of the filters are

$$h_k(t) = \phi_k(T-t), \quad 0 \le t \le T$$



DETECTION THEORY

- Given M possible hypotheses H_i (signal m_i) with probability

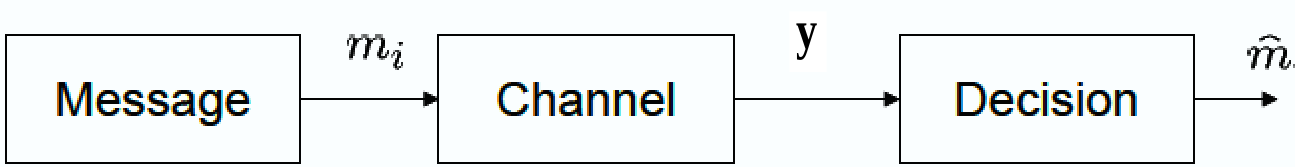
$$P_i = P(m_i) \quad , \quad i = 1, 2, \ldots, M$$

  - P_i represents the prior knowledge concerning the probability of the signal m_i – Prior Probability

- The observation is some collection of N real values, denoted by $\mathbf{y} = (y_1, y_2, \ldots.. y_N)$ with conditional pdf

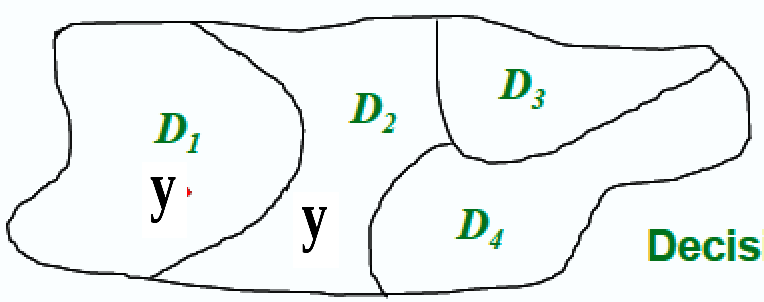
  $f(\mathbf{y} \mid m_i)$ -- conditional pdf of observation $\mathbf{y}$ given the signal m_i

- Goal: Find the best decision-making algorithm in the sense of minimizing the probability of decision error.



DETECTION THEORY
Observation Space

- In general, $\mathbf{y}$ can be regarded as a point in some observation space
- Each hypothesis H_i is associated with a decision region D_i:
- The decision will be in favor of H_i if $\mathbf{y}$ is in D_i
- Error occurs when a decision is made in favor of another when the signals $\mathbf{y}$ falls outside the decision region D_i



Decision Space M=4

DETECTION THEORY
MAP Decision Criterion

- Consider a decision rule based on the computation of the posterior probabilities defined as

$$P(m_i \,|\, \mathbf{y}) = P(\text{ signal } m_i \text{ was transmitted given } \mathbf{y} \text{ observed })$$
$$\text{for } i = 1, \ldots, M$$

  - Known as a posterior since the decision is made after (or given) the observation
  - Different from the a prior where some information about the decision is known in advance of the observation

DETECTION THEORY
MAP Decision Criterion

- By Bayes' Rule: $$P(m_i \mid \mathbf{y}) = \frac{P_i f(\mathbf{y} \mid m_i)}{f(\mathbf{y})}$$
- Since our criterion is to minimize the probability of detection error given $\mathbf{y}$, we deduce that the optimum decision rule is to choose $\hat{m} = m_k$ if and only if $P(m_i \mid \mathbf{y})$ is maximum for $i = k$.
- Equivalently,

$$\boxed{\begin{array}{l} \text{Choose } \hat{m} = m_k \text{ if and only if} \\ P_k f(\mathbf{y} \mid m_k) \geq P_i f(\mathbf{y} \mid m_i) \text{ for all } i \neq k \end{array}}$$

- This decision rule is known as maximum a posterior or MAP decision criterion

DETECTION THEORY
MAP Decision Criterion

- If $p_1 = p_2 = \ldots = p_M$, i.e. the signals $\{m_k\}$ are equiprobable, finding the signal that maximizes $P(m_k \mid \mathbf{y})$ is equivalent to finding the signal that maximizes $f(\mathbf{y} \mid m_k)$
- The conditional pdf $f(\mathbf{y} \mid m_k)$ is usually called the likelihood function. The decision criterion based on the maximum of $f(\mathbf{y} \mid m_k)$ is called the Maximum-Likelihood (ML) criterion.
- ML decision rule:

$$\boxed{\text{Choose } \hat{m} = m_k \text{ if and only if } \quad f(\mathbf{y} \mid m_k) \geq f(\mathbf{y} \mid m_i) \text{ for all } i \neq k}$$

- In any digital communication systems, the decision task ultimately reverts to one of these rules

DETECTION THEORY
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- In any digital communication systems, the decision task ultimately reverts to one of these rules

DETECTION OF BINARY ANTIPODAL SIGNALS IN GAUSSIAN NOISE

During a given signaling interval T, a binary baseband system will transmit one of two waveforms

$$s_i(t) = \begin{cases} s_1(t) & 0 \le \mathrm{t} < \mathrm{T} \quad \text{for a binary 1} \\ \mathrm{s}_2(t) & 0 \le \mathrm{t} < \mathrm{T} \quad \text{for a binary 0} \end{cases}$$

$$\Downarrow$$

$$r(t) = s_i(t) * h_c(t) + n(t) \quad \text{for i} = 1,2$$

At the end of each symbol duration T, the output of the sampler yields a sample $y(T)$ called **test statistic.**

$y(T)$ has a voltage value directly proportional to the energy of the received symbol and that of the noise.

DETECTION FO BINARY ANTIPODAL SIGNALS IN GAUSSIAN NOISE

Let us assume that the input noise w(t) is a random Gaussian process and the receiveing filter is linear

the output noise $n(T) = n_0$ is a zero mean Gaussian random variable

y(T) is a random variable with a mean of either a_1 or a_2 where

$$a_1 = y_o(T) \text{ when } s_1(\mathrm{t}) \text{ is transmitt ed}$$

$$a_2 = y_o(T) \text{ when } s_2(\mathrm{t}) \text{ is transmitt ed}$$

DETECTION FO BINARY ANTIPODAL SIGNALS IN GAUSSIAN NOISE

Conditional pdfs:

$$p(y \mid s_1) = \frac{1}{\sigma_0 \sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{y - a_1}{\sigma_0}\right)^2 \right]$$

$$p(y \mid s_2) = \frac{1}{\sigma_0 \sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{y - a_2}{\sigma_0}\right)^2 \right]$$

DETECTION FO BINARY ANTIPODAL SIGNALS IN GAUSSIAN NOISE

Notes: after a received waveform has been transformed to a sample, the actual shape of the waveform is no longer important; all waveform types that are transformed to the same value of y(T) are identical for detection purposes.
<u>Tha matched filter maps all signals of equal energy into the same point y(T)</u>

The received signal energy (not its shape!) is the important parameter in the detection process.
This is why the detection analysis for baseband signals is the same as that for bandpass
Since y(T) is a voltage signal that is proportional to the energy of the received symbol the larger the magnitue of y(T) the more error free will be the decision process

DETECTION FO BINARY ANTIPODAL SIGNALS IN GAUSSIAN NOISE

Detection is done according to:

$$y(T) > \gamma \Rightarrow H_1$$

$$y(T) < \gamma \Rightarrow H_2$$

H_1, H_2 are the two binary hypothesis

How to choose γ ?

DETECTION FO BINARY ANTIPODAL SIGNALS IN GAUSSIAN NOISE

The likelihood ratio test and the Maximum A Posteriori Criterion

The MAP (Maximum A posteriori criterion) select the threshold that maximize the a posteriori probability, i.e.:

$$\text{choose hypothesis } H_1 \text{ when } p(s_1 \mid y) > p(s_2 \mid y)$$

$$\text{choose hypothesis } H_2 \text{ when } \underbrace{p(s_1 \mid y)}_{} < p(s_2 \mid y)$$

A posteriori probability

Bayes' theorem

$$\text{choose hypothesis } H_1 \text{ when } p(y \mid s_1)P(s_1) > p(y \mid s_2)P(s_2)$$

$$\text{choose hypothesis } H_2 \text{ when } p(y \mid s_1)P(s_1) < p(y \mid s_2)P(s_2)$$

DETECTION FO BINARY ANTIPODAL SIGNALS IN GAUSSIAN NOISE

The likelihood ratio test and the Maximum A Posteriori Criterion

$$\text{choose hypothesis } H_1 \text{ when } \frac{p(y\,|\,s_1)}{p(y\,|\,s_2)} > \frac{P(s_2)}{P(s_1)}$$

$$\text{choose hypothesis } H_2 \text{ when } \underbrace{\frac{p(y\,|\,s_1)}{p(y\,|\,s_2)}} < \frac{P(s_2)}{P(s_1)}$$

Likelihood ratio test

If errors are uniformly distributed, this MAP criterion corresponds to a Minimum error criterion (on the average, it yields the minimum number of incorrect decision)

DETECTION FO BINARY ANTIPODAL SIGNALS IN GAUSSIAN NOISE

Maximum likelihood Criterion

When the classes are equally likely (in case of two classes ⇨ $P(s_1) = P(s_2)$)
The MAP criterion is known as **Maximum likelihood criterion**

$$\text{choose hypothesis } H_1 \text{ when } \frac{p(y \mid s_1)}{p(y \mid s_2)} > 1$$

$$\text{choose hypothesis } H_2 \text{ when } \frac{p(y \mid s_1)}{p(y \mid s_2)} < 1$$

DETECTION FO BINARY ANTIPODAL SIGNALS IN GAUSSIAN NOISE

Maximum likelihood Criterion

In case of binary decision, the optimum threshold is:

$$\gamma_0 = \frac{a_1 + a_2}{2}$$

For antipodal signals $a_1 = -a_2$ $\Rightarrow$ $\gamma_0 = 0$

DETECTION FO BINARY ANTIPODAL SIGNALS IN GAUSSIAN NOISE

Error Probability

$$a_1 = -a_2 = \sqrt{E_b} \qquad \gamma = 0$$

$$P_e = P(s_1)P(y<0|s_1) + P(s_2)P(y>0|s_2) =$$
$$= P(s_1)P(\sqrt{E_b}+n<0) + P(s_2)P(-\sqrt{E_b}+n>0)$$

$$P(s_1) = P(s_2) = 1/2$$

$$P(\sqrt{E_b}+n<0) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{0} e^{-\frac{1}{2}\left(\frac{v-\sqrt{E_b}}{\sigma}\right)^2} dv$$

variable change

$$P(\sqrt{E_b}+n<0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\sqrt{\frac{2E_b}{N_0}}} e^{-\frac{1}{2}x^2} dv = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

$$P(-\sqrt{E_b}+n>0) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

$$\boxed{P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)}$$

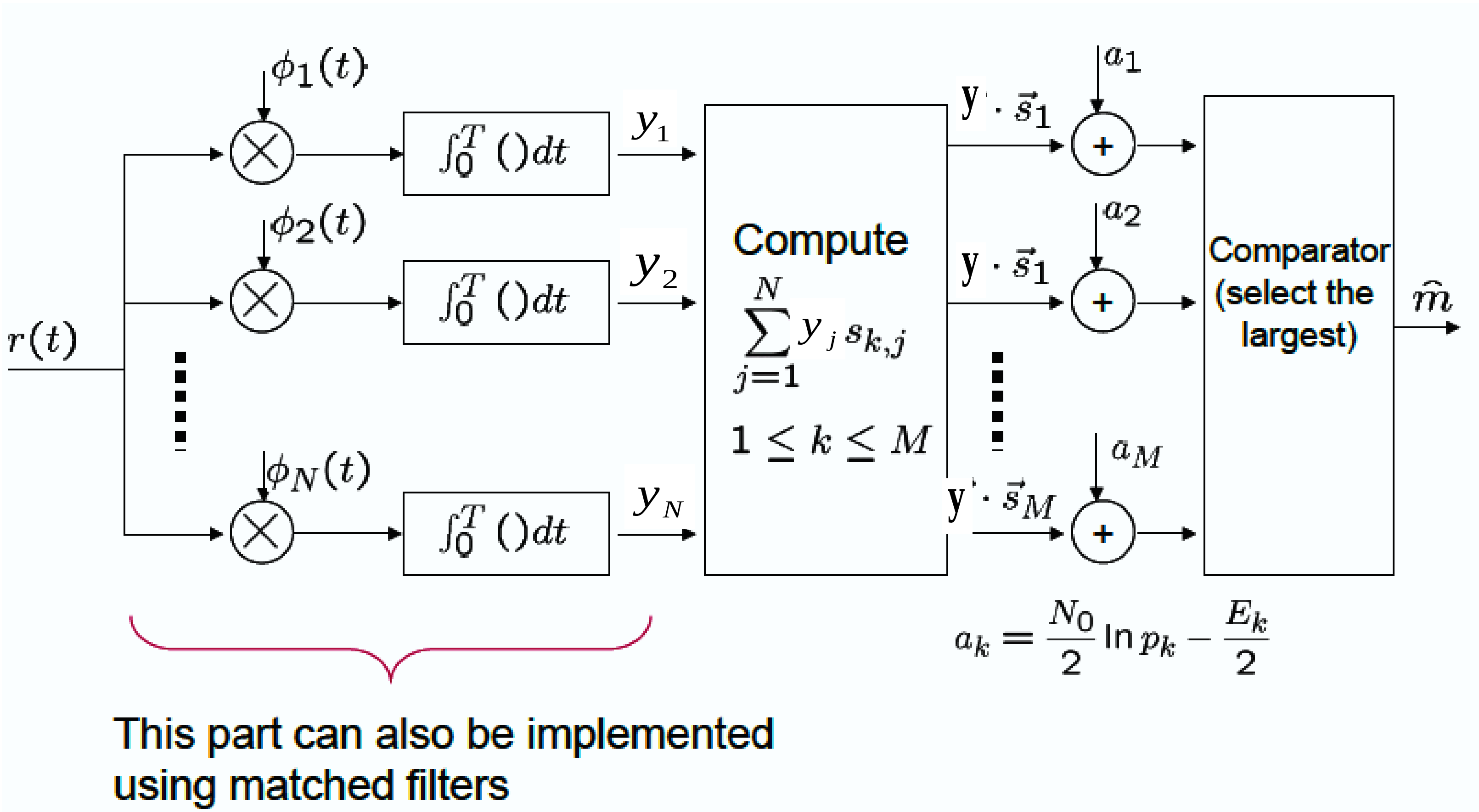
DETECTION OF BINARY SIGNALS IN GAUSSIAN NOISE

Error Probability

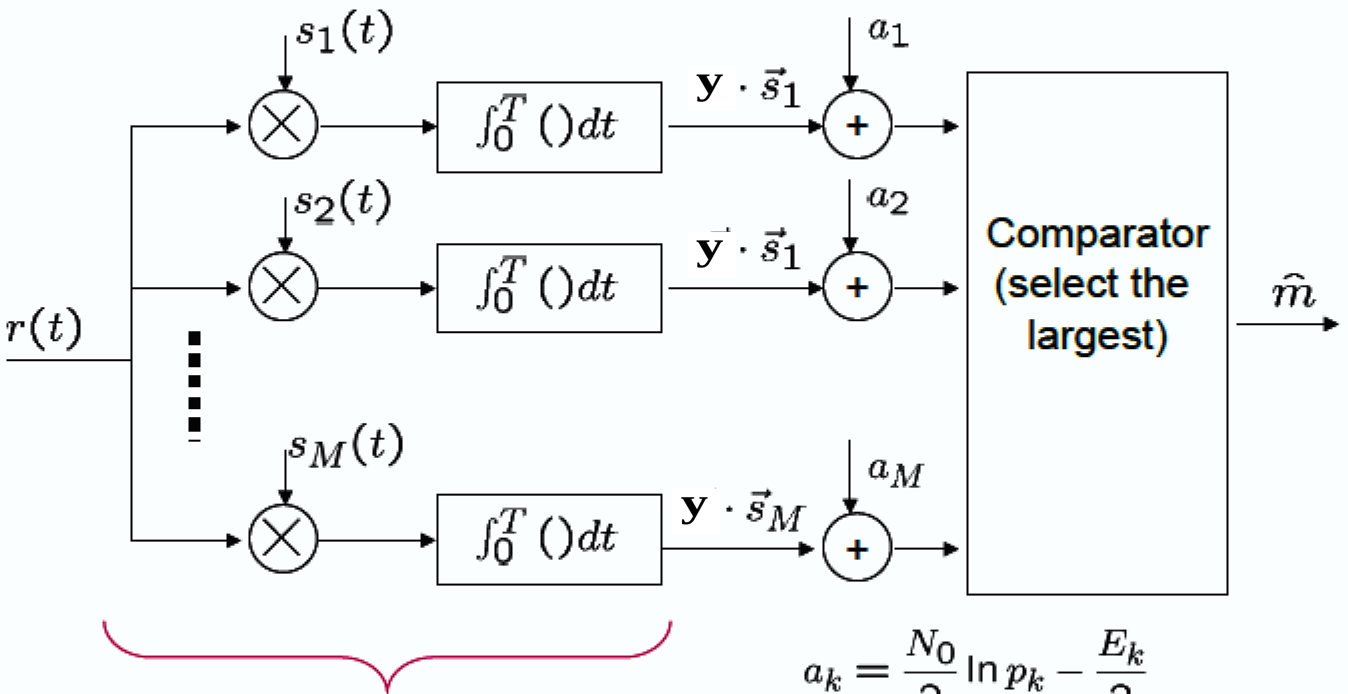
Prove the in case of binary ORTHOGONAL signals the probability of error is

$$\boxed{P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)}$$

MAP receiver structure
Method #1 (signal demodulator + detector)



MAP receiver structure
Method #2 (integrated demodulator + detector)



$$a_k = \frac{N_0}{2} \ln p_k - \frac{E_k}{2}$$

This part can also be implemented using matched filters

$$\hat{m} = \arg \max_{m_1,\dots,m_M} \left\{ \mathbf{y} \cdot \vec{s}_k - \frac{E_k}{2} + \frac{N_0}{2} \ln P_k \right\}$$

MAP receiver structure
Method #1 vs Method #2

- Both receivers perform identically
- Choice depends on circumstances
- For instance, if N < M and $\{\phi_j(t)\}$ are easier to generate than $\{s_k(t)\}$, then the choice is obvious

Decision Regions

- Signal space can be divided into M disjoint decision regions R_1 R_2, …, R_M.

$$\boxed{\text{If } \vec{\mathbf{y}} \in R_k \Longrightarrow \text{decide } m_k \text{ was transmitted}}$$

Select decision regions so that P_e is minimized

- Recall that the optimal receiver sets $\hat{m} = m_k$ iff

  $\|\mathbf{y} - \vec{s}_k\|^2 - N_0 \ln P_k$ is minimized

- For simplicity, if one assumes $p_k = 1/M$, for all k, then the optimal receiver sets $\hat{m} = m_k$ iff

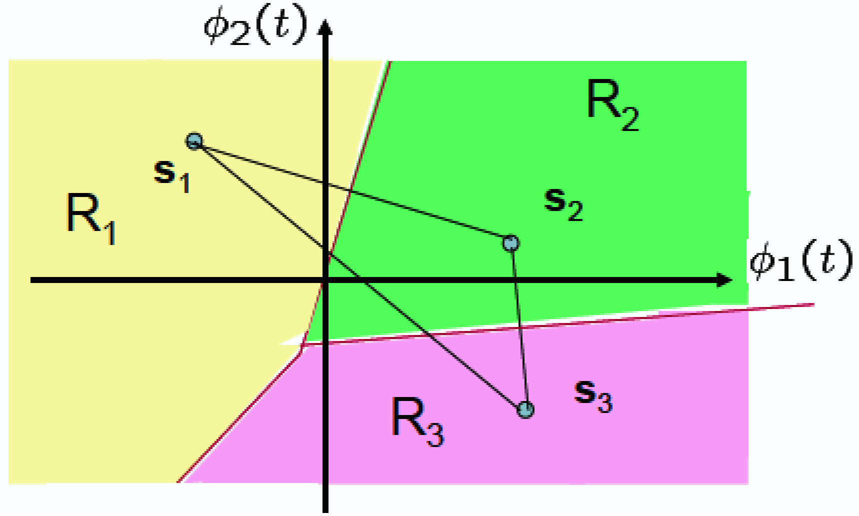
  $\|\mathbf{y} - \vec{s}_k\|^2$ is minimized

Decision Regions

- Geometrically, this means
  - Take projection of r(t) in the signal space (i.e. $\vec{\mathbf{y}}$). Then, decision is made in favor of signal that is the closest to $\mathbf{y}$ in the sense of minimum Euclidean distance
  - And those observation vectors $\mathbf{y}$ with $\|\mathbf{y}-\mathbf{s}_\mathbf{k}\|^2 < \|\mathbf{y}-\mathbf{s}_i\|^2$ for all $i \neq k$ should be assigned to decision region R_k

Decision Regions

- In general, boundaries of decision regions are perpendicular bisectors of the lines joining the original transmitted signals
- Example: three equiprobable 2-dim signals



Decision Regions
Example: QPSK modulation

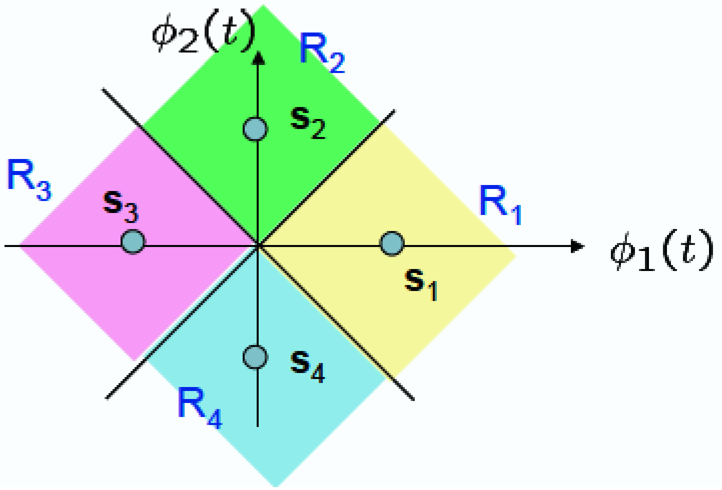
- Assume all signals are equally likely
- All 4 signals could be written as the linear combination of two basis functions
- Constellations of 4 signals

$s_1=(1,0)$

$s_2=(0,1)$

$s_3=(-1,0)$

$s_4=(0,-1)$



Decision Regions

- Boundaries are perpendicular to a line drawn between two signal points
- If signals are equiprobable, decision boundaries lie exactly halfway in between signal points
- If signal probabilities are unequal, the region of the less probable signal will shrink

Decision Regions

Three equally probable messages m1, m2, and m3 are to be transmitted over an AWGN channel with noise power-spectral density $N_0 / 2$. The messages are

$$s_1(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & otherwise \end{cases}$$

$$s_2(t) = -s_3(t) = \begin{cases} 1 & 0 \le t \le \frac{T}{2} \\ -1 & \frac{T}{2} \le t \le T \\ 0 & otherwise \end{cases}$$

1. What is the dimensionality of the signal space ?
2. Find an appropriate basis for the signal space (Hint: You can find the basis without using the Gram-Schmidt procedure).
3. Draw the signal constellation for this problem.
4. Sketch the optimal decision regions R1, R2, and R3.
5. Which of the three messages is more vulnerable to errors and why ? In other words, which of $p(Error \mid m_i \quad transmitted), \ i = 1, \ 2, \ 3$ is larger ?

Union Bound on the Probability of Error

For binary equiprobable signaling over an AWGN channel, regardless of the signaling type, the error probability can be expressed as:

$$\boxed{P_e = Q\left(\frac{d}{\sqrt{2N_0}}\right)}$$

Where d is the Euclidean distance between the two signal points in the constellation

Does it exist a simple expression for the error probability of the general equiprobable M-ary signaling?

NO

$$\boxed{P_M = \frac{1}{M}\sum_{i=1}^{M} P(e \mid s_m) = \frac{1}{M}\sum_{i=1}^{M} \int_{R_m^c} f(\mathbf{y} \mid s_m) d\mathbf{y} = \frac{1}{M}\sum_{i=1}^{M} \left(1 - \int_{R_m} f(\mathbf{y} \mid s_m) d\mathbf{y}\right) =}$$

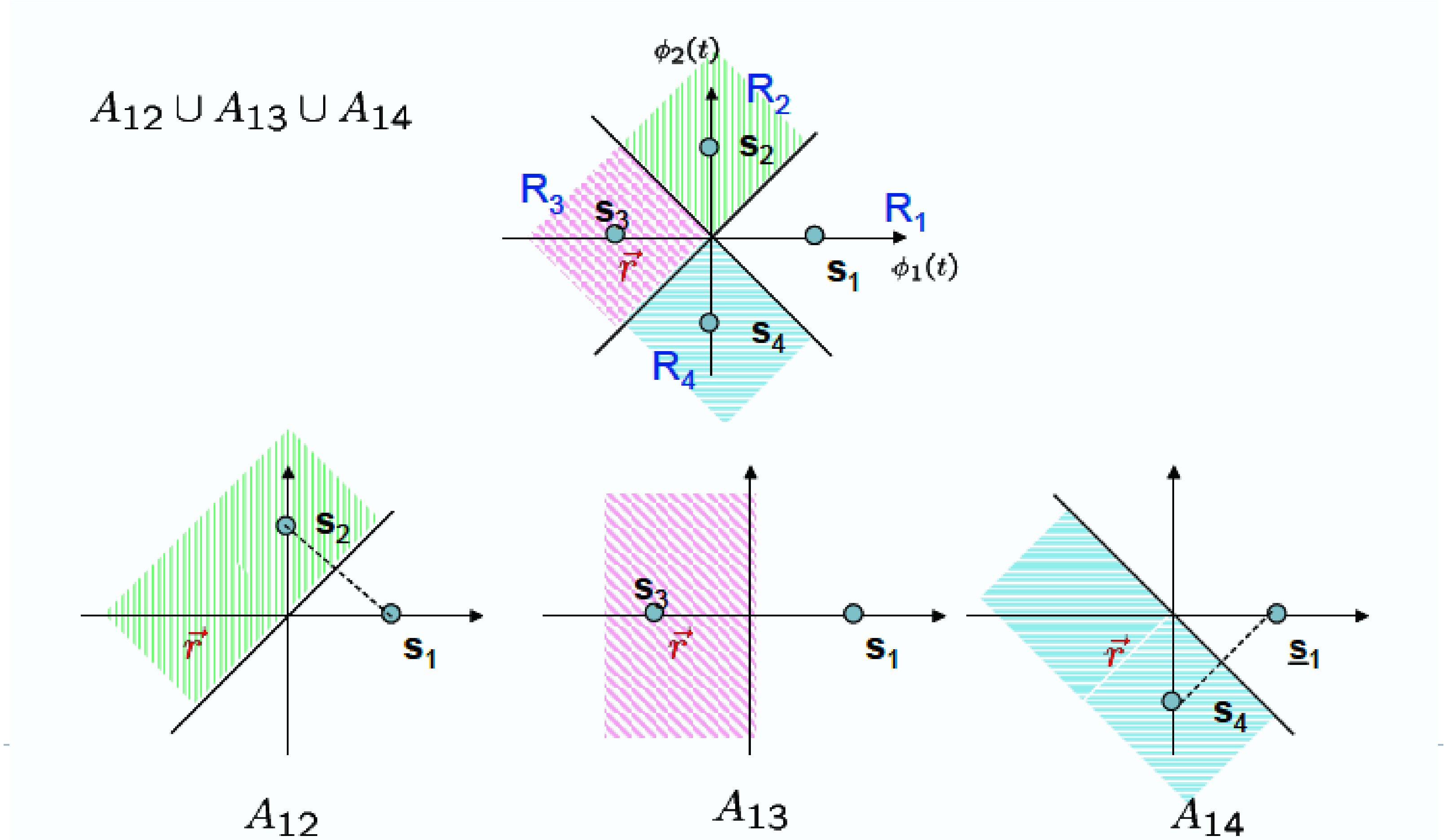
However, there is a simple upper bound known as *UNION BOUND*

Union Bound on the Probability of Error

Let us assume that $s_m(t)$ is transmitted

The proability of error is the probability that the receiver detects a signal other than $s_m(t)$

Let A_i the event that message i is detected at the receiver



Union Bound on the Probability of Error

$$P_m = P(error \mid s_m(t) \text{ sent}) = P\left(\bigcup_{\substack{i=1 \\ i \neq m}}^{M} A_i \mid s_m(t) \text{ sent} \right) \leq \sum_{\substack{i=1 \\ i \neq m}}^{M} P(A_i \mid s_m(t) \text{ sent})$$

A necessary BUT not sufficient condition for detecting $s_i(t)$ when $s_m(t)$ is sent (if the ML criterion is used):

$$D(\mathbf{y}, s_i) < D(\mathbf{y}, s_m)$$

$$\Downarrow$$

$$P(A_i \mid s_m(t) \text{ sent}) \leq P(D(\mathbf{y}, s_i) < D(\mathbf{y}, s_m))$$

$$\Downarrow$$

$$P(D(\mathbf{y}, s_i) < D(\mathbf{y}, s_m)) = Q\left(\frac{d_{mi}}{\sqrt{2N_0}} \right)$$

Union Bound on the Probability of Error

$$P_m \le \sum_{\substack{i=1 \\ i\neq m}}^{M} P(A_i \mid s_m(t)\ \text{sent}) \le \sum_{\substack{i=1 \\ i\neq m}}^{M} Q\left(\frac{d_{mi}}{\sqrt{2N_0}}\right)$$

If we define

$$d_{\min} = \min_{\substack{1\le m,m' \le M \\ m' \neq m}} d_{mm'}$$

$$Q\left(\frac{d_{mi}}{\sqrt{2N_0}}\right) \le Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

$$P_m \le \sum_{\substack{i=1 \\ i\neq m}}^{M} Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = (M-1)Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

Union Bound on the Probability of Error

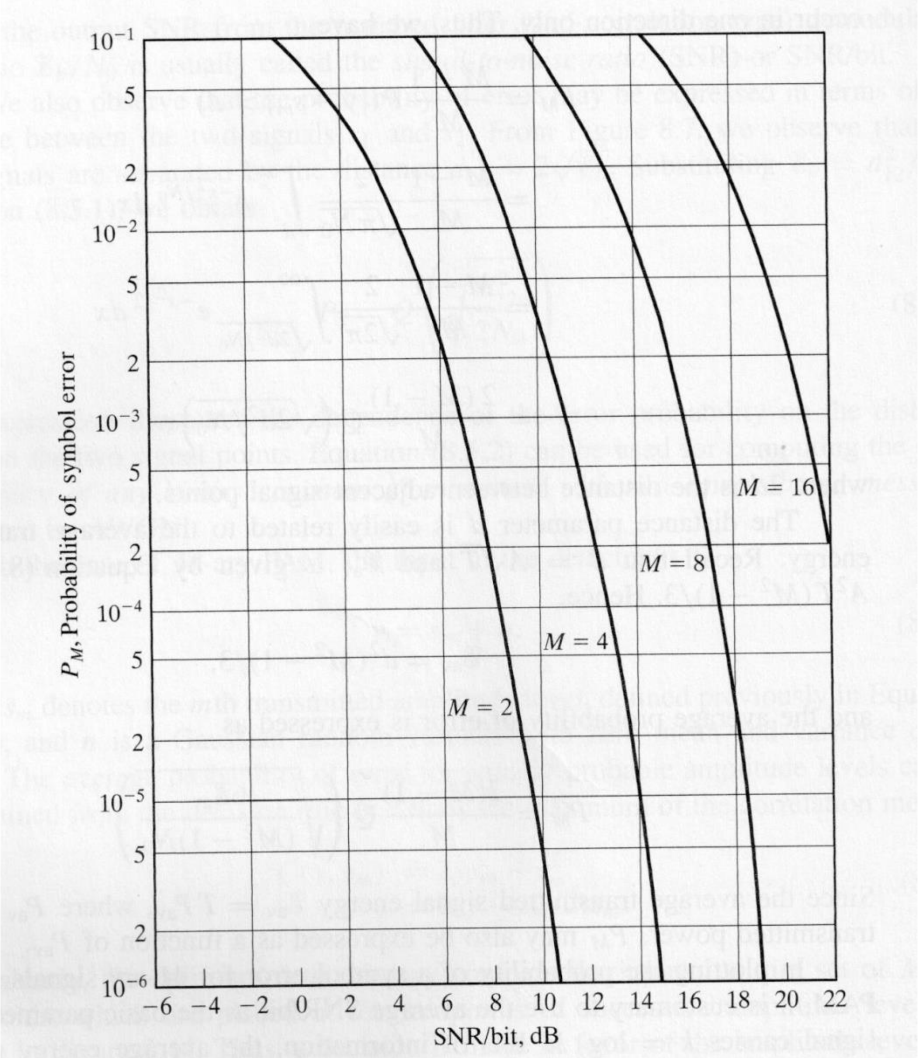
$$\boxed{P_M = \frac{1}{M}\sum_{\substack{i=1\\ i\neq m}}^{M} P_m \le (M-1)Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) \le \frac{M-1}{2}e^{-\frac{d_{\min}^2}{4N_0}}}$$

$$Q(x) \le 1/e^{-\frac{x^2}{2}}$$

Thi approximation is very useful for high SNR

A good signal set should provide the maximum possible $d_{\min}$

Probability of Error for M-ary PAM



Probability of Error for M-ary orthogonal signaling