Tutorial, Session 3. May, 07

Q. 1- What is the resulting SNR for a signal uniformly distributed on [-1, 1] when uniform PCM with 256 levels is employed.

sol.

$$SNR = \frac{\sigma_x^2}{\sigma_n^2}, \quad \sigma_n^2 = \frac{\nabla^2}{12}, \quad \nabla = \frac{x_{max}}{2^{\nu-1}}$$

$$x_{max} = 1, \quad \nu = \log_2 256 = 8,$$

There fore:

$$\nabla = \frac{1}{2^7} = 0.0078125$$

$$\sigma_n^2 = \cancel{0.0078125}\ 5.1 \times 10^{-6}$$

$$\sigma_x^2 = \frac{2^2}{12} = \frac{1}{3}$$

$$SNR = \frac{\sigma_x^2}{\sigma_n^2} \simeq 65359 \sim 48.16\,dB.$$

Q. 2 (6.55 Prokis-Salehi)

The random process $X(t)$ is defined by $X(t) = Y \cos(2\pi f_0 t + \Theta)$ where Y and Θ are two independent random variables, Y uniform on $[-3, 3]$ and Θ uniform on $[0, 2\pi]$.

1. Find the autocorrelation function of $X(t)$ and its Power-spectral density.

2. If $X(t)$ is to be transmitted to maintain a SNR of at least 40 dB using a uniform PCM system, what is the required number of bits/sample and the least bandwidth requirement (in terms of f_0)?

3. If the SNR is to be increased by 24 dB, how many more bits/sample have to be introduced and what is the new minimum bandwidth requirement in this case?

sol.

1. $x(t) = y \cos(2\pi f_0 t + \theta)$, Y, θ are independent

$$Y \sim U[-3,3] \longrightarrow f_Y(y) = \begin{cases} \frac{1}{6} & 3 \le y \le 3 \\ 0 & \text{e.w.} \end{cases}$$

$$\theta \sim U[0, 2\pi] \longrightarrow f_\theta(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \le \theta \le 2\pi \\ 0 & \text{e.w.} \end{cases}$$

$R_X(t_1, t_2) = ?$

$$R_X(t_1, t_2) = E[x(t_1)\,x(t_2)] = E[\,Y \cos(2\pi f_0 t_1 + \theta)\, Y \cos(2\pi f_0 t_2 + \theta)]$$

$$= E\left[Y^2 \cdot \frac{1}{2}\left\{\cos(2\pi f_0 (t_2 - t_1)) - \cos(2\pi f_0 (t_1 + t_2) + 2\theta)\right\}\right]$$

$$= \frac{1}{2}\underbrace{E[Y^2]}_{3}\left\{E[\cos(2\pi f_0 \underbrace{(t_2 - t_1)}_{\tau})] - \cancelto{0}{E[\cos(2\pi f_0 (t_1 + t_2) + 2\theta)]}\right\}$$

$$= \frac{3}{2}\cos 2\pi f_0 \tau$$

$$R_X(\tau) = \frac{3}{2}\cos 2\pi f_0 \tau \xleftrightarrow{\ F\ } G_X(f) = \frac{3}{4}\left[\delta(f - f_0) + \delta(f + f_0)\right]$$

2. $SNR = 40\ dB$

$v = ?\ ,\quad \min B = ?$

$$SNR = \frac{\sigma_x^2}{\sigma_n^2}\ ,\quad \sigma_x^2 = R_x(0) = \frac{3}{2}$$

$$\rightarrow \sigma_n^2 = \frac{1.5}{10^4} = 1.5\times 10^{-4}$$

$$\sigma_n^2 = \frac{\Delta^2}{12}$$

$$\Delta = \frac{x_{max}}{2^{v-1}} \longrightarrow \sigma_n^2 = \frac{x_{max}^2}{3\cdot 4^{v}} \longrightarrow v = \frac{1}{2}\log_2 \frac{x_{max}^2}{3\cdot \sigma_n^2}$$

$$v = \frac{1}{2}\log_2 \frac{3^2}{3\times 1.5\times 10^{-4}} = \frac{1}{2}\log_2 2\times 10^{+4} = \lceil 7.1439 \rceil = 8$$

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$$R = \underbrace{f_s}_{\text{sampling frequeny}} \cdot v \geq 2\cdot f_0 \cdot v = 16 f_0$$

$$\text{minimum bandwidth} = \frac{16 f_0}{2} = 8 f_0 .$$

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3. $SNR = 40 + 24\ dB = 64\ dB$

we know that: $SNR \sim 6v \longrightarrow$ we need 4 more bits

$v_{new} = 8 + 4 = 12 .$

$$B = \frac{2\cdot f_0 \cdot v}{2} = 12 f_0 .$$

Q.3 (6.53 Prokis-Salehi)

A signal can be modeled as a lowpass stationary process $X(t)$ whose PDF at any time t_0 is given in Figure P-6.53

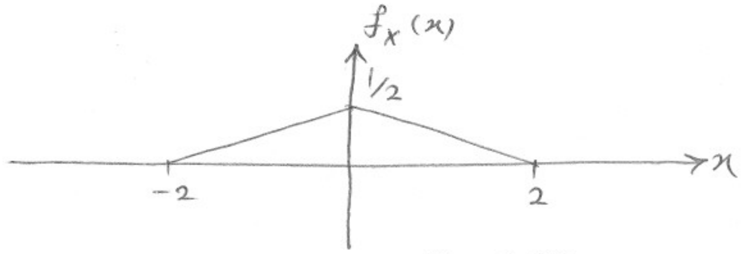

Figure P_6.53

The bandwidth of this process is 5 KHz, and it is desired to transmit it using a PCM system.

1. If sampling is done at the Nyquist rate and a uniform quantizer with 32 levels is employed, what is the resulting SNR? what is the resulting bit rate?

2. If the available bandwidth of the channel is 40 KHz, what is the highest achievable SNR?

3. If instead of sampling at the Nyquist rate we require a guard-band of at least 2 KHz and the bandwidth of the channel is 40 KHz, what is the highest achievable SNR?

sol.

1. $f_s = 2W = 2\times 5 = 10\ KHZ.$ $\left.\begin{array}{l} \\ \end{array}\right\} \longrightarrow SNR = ?$

$v = \log_2 32 = 5$ bit rate $= ?$

$$SNR = \frac{\sigma_x^2}{\sigma_n^2}.$$

$$\sigma_x^2 = E[x^2] - \cancel{(E[x])^2}^{\,0} = E[x^2] = \int_{-\infty}^{\infty} x^2 f_x(x)\,dx$$

$$= \int_{-2}^{0} (\tfrac{1}{4}x + 1/2)\, x^2\,dx + \int_0^2 (-\tfrac{1}{4}x + \tfrac{1}{2})\, x^2\,dx$$

$$= 2\times\int_0^2 x^2(-\tfrac{1}{4}x + \tfrac{1}{2})\,dx = \int_0^2 (-\tfrac{1}{2}x^3 + x^2)\,dx$$

$$= \left(-\tfrac{1}{8}x^4 + \tfrac{1}{3}x^3\right)_0^2 = -\tfrac{1}{8}\cdot 16 + \tfrac{1}{3}\cdot 8 = \tfrac{2}{3} \quad *1$$

$$\sigma_n^2 = \frac{\Delta^2}{12}, \quad \Delta = \frac{x_{max}}{2^{v-1}} = \frac{2}{2^4} = \frac{1}{8} = 0.125$$

$$\sigma_n^2 = 0.0013 \quad *2$$

$$*1,2 \longrightarrow SNR = \frac{2}{3\times 0.0013} = 512 \simeq 27.1\ dB$$

bit rate $= f_s \cdot v = 50\ KHZ.$

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2. $vW = B \longrightarrow 5\times 5\,KHZ = 25\ KHZ < 40\ KHZ$

$v_{new}\cdot W = B \longrightarrow v_{new} = \frac{40}{5} = 8$ bit / sample

$SNR_{new} = 27.1 + 3\times 6 = 45.1\ dB$

3. $f_s = 12\ KHZ$

$R = \nu \cdot 12 \longrightarrow$ ~~Require~~ Minimum Bandwidth $= 6 \cdot \nu$

$6 \cdot \nu = 40\ KHZ \longrightarrow \nu = \left\lceil \frac{40}{6} \right\rceil = 6$ bit/sample.

$SNR_{new} = 27.1 + 6 = 33.1\ dB.$