ECE 342 Communication Theory
Fall 2005, Homework 1

Prof. Tiffany J Li
`www: http://www.eecs.lehigh.edu/~jingli/teach`
`email: jingli@ece.lehigh.edu`

Given: Wednesday Sept 28 5:35-6:55pm   In class, closed book, 1 letter-size sheet of notes.

---

1. (20 points) Determine whether the following statements are true or false. If false, provide a correct statement or a justification.

   (a) (T) Any signal that is a function of time has a frequency-domain representation that can be obtained using Fourier Transform.

   (b) (F) The amplitude spectrum (obtained by the Fourier Transform) of a signal is even symmetric while the phase spectrum is odd symmetric. **(This holds for real signals only.)**

   (c) (T) The Fourier transform of a periodic signal consists of a sequence of impulses in frequency at multiples of the fundamental frequency of the periodic signal.

   (d) (T) If $x(t) \Longleftrightarrow X(f)$, and $y(t) \Longleftrightarrow Y(f)$, then $x(t)y(t) \Longleftrightarrow X(f) * Y(f)$ (i.e. multiplication in time translates to convolution in frequency).

   (e) (F) In a DSB-SC signal, the envelope of the resulting bandpass signal is proportional to the amplitude of the message signal. **(The envelope of the resulting bandpass signal is proportional to the absolute value of the message signal's amplitude.)**

---

2. (10 points) Find the trigonometric Fourier series and sketch the corresponding spectra for the periodic impulse train $g(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$.

   Solution: $f_0 = 1/T_0$ and $w_0 = 2\pi/T_0$. The trigonometric Fourier series for $g(t)$ is given by

   $$g(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(nw_0t) + b_n \sin(nw_0t)).$$

   Since $g(t)$ is even symmetric, its trigonometric Fourier series expansion will not contain $\sin$ terms, i.e. $b_n = 0$. According to the definition:

   $$\begin{aligned} a_0 &= \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)dt = \frac{2}{T_0} \\ a_n &= \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\cos(nw_0t)dt = \frac{2}{T_0} \end{aligned}$$

Hence, we have

$$g(t) = \frac{1}{T_0} + \frac{2}{T_0}\sum_{n=1}^{\infty} cos(nw_0 t) = \frac{2}{T_0} + \frac{2}{T_0}\sum_{n=1}^{\infty} cos(\frac{2n\pi}{T_0}t)$$

Sketch of the spectrum omitted.

---

3. (20 points) Consider a signal

$$x(t) = e^{-\alpha t}u_{-1}(t), \quad \alpha > 0$$

and a linear tie invariant system with response

$$h(t) = sinc(6t).$$

(a) Determine whether a signal $x(t)$ is energy-type or power-type. In each case, find the energy or power-spectral density and also the energy or power content of $x(t)$.

(b) Find the energy spectral density and the energy content, or power-spectral density and the power content of the output of the LTI system $h(t)$ when driven by $x(t)$.

Solution:

(a) $x(t) = e^{-\alpha t}u_{-1}(t)$. The spectrum of the signal is $X(f) = \frac{1}{\alpha + j2\pi f}$ and the energy spectral density

$$\mathcal{G}_X(f) = |X(f)|^2 = \frac{1}{\alpha^2 + 4\pi^2 f^2}$$

Thus, the time correlation of the signal is

$$R_X(\tau) = \mathcal{F}^{-1}[\mathcal{G}_X(f)] = \frac{1}{2\alpha}e^{-\alpha|\tau|}$$

The energy content of the signal is

$$E_X = R_X(0) = \frac{1}{2\alpha} < \infty$$

The signal is energy-type.

(b)

$$h(t) = \text{sinc}(6t) \Longrightarrow H(f) = \frac{1}{6}\Pi(\frac{f}{6})$$

The energy spectral density of the output signal is

$$\mathcal{G}_Y(f) = \mathcal{G}_X(f)|H(f)|^2 = |X(f)|^2|H(f)|^2.$$

With $|X(f)|^2 = \frac{1}{\alpha^2+4\pi^2 f^2}$, we obtain

$$\mathcal{G}_Y(f) = \frac{1}{\alpha^2 + 4\pi^2 f^2}\frac{1}{36}\Pi^2(\frac{f}{6}) = \frac{1}{36(\alpha^2 + 4\pi^2 f^2)}\Pi(\frac{f}{6})$$

The energy content of the output signal is

$$\begin{aligned} E_Y &= \int_{-\infty}^{\infty} \mathcal{G}_Y(f)df = \frac{1}{36}\int_{-3}^{3} \frac{1}{\alpha^2 + 4\pi^2 f^2} df \\ &= \frac{1}{36(2\alpha\pi)} \arctan(f\frac{2\pi}{\alpha})\Big|_{-3}^{3} \\ &= \frac{1}{36\alpha\pi} \arctan(\frac{6\pi}{\alpha}) \end{aligned}$$

---

4. (10 points) In a DSB SC system, the message signal is $m(t) = sinc(t) + sinc^2(t)$ and the carrier is $c(t) = A\cos(2\pi f_c t)$. Find the frequency domain representation and the bandwidth of the modulated signal.

Solution:

$$u(t) = m(t)c(t) = A(\text{sinc}(t) + \text{sinc}^2(t))\cos(2\pi f_c t)$$

Taking the Fourier transform of both sides, we obtain

$$\begin{aligned} U(f) &= \frac{A}{2}\left[\Pi(f) + \Lambda(f)\right] \star (\delta(f - f_c) + \delta(f + f_c)) \\ &= \frac{A}{2}\left[\Pi(f - f_c) + \Lambda(f - f_c) + \Pi(f + f_c) + \Lambda(f + f_c)\right] \end{aligned}$$

$\Pi(f - f_c) \neq 0$ for $|f - f_c| < \frac{1}{2}$, whereas $\Lambda(f - f_c) \neq 0$ for $|f - f_c| < 1$. Hence, the bandwidth of the modulated signal is 2.

---

5. (20 points) In a DSB SC system, the message signal is $m(t) = 2\cos(400t) + 4\sin(500t + \pi/3)$ and the carrier is $c(t) = A\cos(8000\pi t)$.

   (a) Find the time domain and frequency domain representation of the modulated signal and plot the spectrum (Fourier transform) of the modulated signal.

   (b) Find the power content of the modulated signal.

Solution:

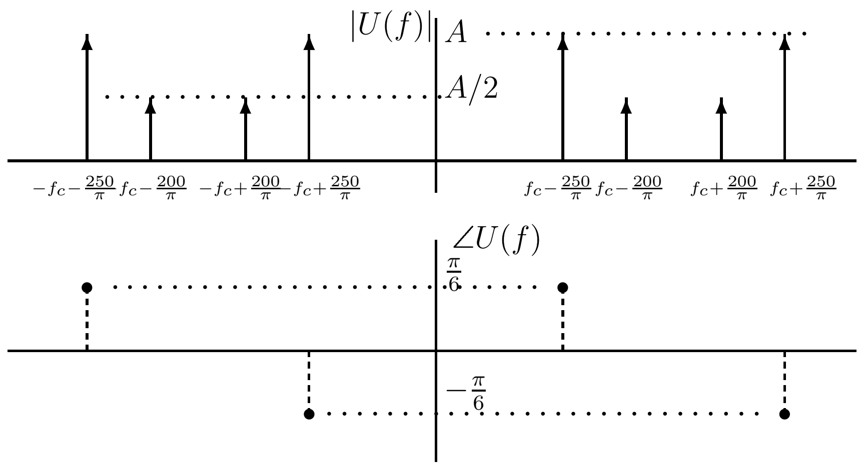
(a) The modulated signal is

$$\begin{aligned} u(t) &= m(t)c(t) = Am(t)\cos(2\pi 4 \times 10^3 t) \\ &= A\left[2\cos(2\pi\frac{200}{\pi}t) + 4\sin(2\pi\frac{250}{\pi}t + \frac{\pi}{3})\right]\cos(2\pi 4 \times 10^3 t) \\ &= A\cos(2\pi(4 \times 10^3 + \frac{200}{\pi})t) + A\cos(2\pi(4 \times 10^3 - \frac{200}{\pi})t) \\ &\quad +2A\sin(2\pi(4 \times 10^3 + \frac{250}{\pi})t + \frac{\pi}{3}) - 2A\sin(2\pi(4 \times 10^3 - \frac{250}{\pi})t - \frac{\pi}{3}) \end{aligned}$$

Taking the Fourier transform of the previous relation, we obtain

$$\begin{aligned}
U(f) &= A\left[\delta(f-\frac{200}{\pi})+\delta(f+\frac{200}{\pi})+\frac{2}{j}e^{j\frac{\pi}{3}}\delta(f-\frac{250}{\pi})-\frac{2}{j}e^{-j\frac{\pi}{3}}\delta(f+\frac{250}{\pi})\right] \\
&\quad \star\frac{1}{2}[\delta(f-4\times 10^3)+\delta(f+4\times 10^3)] \\
&= \frac{A}{2}\Big[\delta(f-4\times 10^3-\frac{200}{\pi})+\delta(f-4\times 10^3+\frac{200}{\pi}) \\
&\quad +2e^{-j\frac{\pi}{6}}\delta(f-4\times 10^3-\frac{250}{\pi})+2e^{j\frac{\pi}{6}}\delta(f-4\times 10^3+\frac{250}{\pi}) \\
&\quad +\delta(f+4\times 10^3-\frac{200}{\pi})+\delta(f+4\times 10^3+\frac{200}{\pi}) \\
&\quad +2e^{-j\frac{\pi}{6}}\delta(f+4\times 10^3-\frac{250}{\pi})+2e^{j\frac{\pi}{6}}\delta(f+4\times 10^3+\frac{250}{\pi})\Big]
\end{aligned}$$

The next figure depicts the magnitude and the phase of the spectrum $U(f)$.



(b) To find the power content of the modulated signal we write $u^2(t)$ as

$$\begin{aligned}
u^2(t) &= A^2\cos^2(2\pi(4\times 10^3+\frac{200}{\pi})t)+A^2\cos^2(2\pi(4\times 10^3-\frac{200}{\pi})t) \\
&\quad +4A^2\sin^2(2\pi(4\times 10^3+\frac{250}{\pi})t+\frac{\pi}{3})+4A^2\sin^2(2\pi(4\times 10^3-\frac{250}{\pi})t-\frac{\pi}{3}) \\
&\quad +\text{terms of cosine and sine functions in the first power}
\end{aligned}$$

Hence,

$$P=\lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}}u^2(t)dt=\frac{A^2}{2}+\frac{A^2}{2}+\frac{4A^2}{2}+\frac{4A^2}{2}=5A^2$$

---

6. (20 points) A SSB AM signal is generated by modulating an 800-kHz carrier by the signal $m(t)=cos(2000\pi t)+2\sin(2000\pi t)$. The amplitude of the carrier is $A_c=100$.

   (a) Determine the $\hat{m}(t)$, the Hilbert transform of $m(t)$.

(b) Determine the time domain expression for the lower sideband of the SSB AM signal.

(c) Determine the magnitude spectrum of the lower sideband SSB signal.

Solution:

(a) The Hilbert transform of $\cos(2\pi 1000t)$ is $\sin(2\pi 1000t)$, whereas the Hilbert transform of$\sin(2\pi 1000t)$ is $-\cos(2\pi 1000t)$. Thus

$$\hat{m}(t) = \sin(2\pi 1000t) - 2\cos(2\pi 1000t)$$

(b) The expression for the LSSB AM signal is

$$u_l(t) = A_c m(t)\cos(2\pi f_c t) + A_c \hat{m}(t)\sin(2\pi f_c t)$$

Substituting $A_c = 100$, $m(t) = \cos(2\pi 1000t) + 2\sin(2\pi 1000t)$ and $\hat{m}(t) = \sin(2\pi 1000t) - 2\cos(2\pi 1000t)$ in the previous, we obtain

$$\begin{aligned}
u_l(t) &= 100\left[\cos(2\pi 1000t) + 2\sin(2\pi 1000t)\right]\cos(2\pi f_c t) \\
&+ 100\left[\sin(2\pi 1000t) - 2\cos(2\pi 1000t)\right]\sin(2\pi f_c t) \\
&= 100\left[\cos(2\pi 1000t)\cos(2\pi f_c t) + \sin(2\pi 1000t)\sin(2\pi f_c t)\right] \\
&+ 200\left[\cos(2\pi f_c t)\sin(2\pi 1000t) - \sin(2\pi f_c t)\cos(2\pi 1000t)\right] \\
&= 100\cos(2\pi(f_c - 1000)t) - 200\sin(2\pi(f_c - 1000)t)
\end{aligned}$$

(c) Taking the Fourier transform of the previous expression we obtain

$$\begin{aligned}
U_l(f) &= 50\left(\delta(f - f_c + 1000) + \delta(f + f_c - 1000)\right) \\
&+ 100j\left(\delta(f - f_c + 1000) - \delta(f + f_c - 1000)\right) \\
&= (50 + 100j)\delta(f - f_c + 1000) + (50 - 100j)\delta(f + f_c - 1000)
\end{aligned}$$

Hence, the magnitude spectrum is given by

$$\begin{aligned}
|U_l(f)| &= \sqrt{50^2 + 100^2}\left(\delta(f - f_c + 1000) + \delta(f + f_c - 1000)\right) \\
&= 10\sqrt{125}\left(\delta(f - f_c + 1000) + \delta(f + f_c - 1000)\right)
\end{aligned}$$